

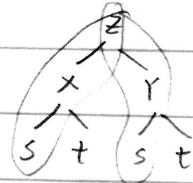
# Midterm 2 Review Oct. 10

## Topic 1. Partial derivatives and chain rule.

### ① Partial derivatives. $f(x, y)$

$f_x$  or  $\frac{\partial f}{\partial x}$ : treat  $y$  as a constant and take the usual derivative with respect to  $x$   
 $f_y$  or  $\frac{\partial f}{\partial y}$ : treat  $x$  as a constant and take the usual derivative with respect to  $y$   
 $f_{xx} = (f_x)_x$

### ② Chain Rule: $z = f(x, y)$ , $x = x(s, t)$ , $y = y(s, t)$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

### ③ Implicit differentiation.

Given  $F(x, y) = 0$ , derivative  $\frac{dy}{dx} = -\frac{F_x}{F_y}$

Remark:

Thm: If  $f_x, f_y$  exist around  $(a, b)$  and are continuous at  $(a, b)$ , then  $f$  is differentiable at  $(a, b)$

Thm: If  $f$  is differentiable at  $(a, b)$ , then  $f$  is continuous at  $(a, b)$

If  $f$  is not continuous at  $(a, b)$ , then  $f$  is not differentiable at  $(a, b)$

Examples

P993 #45  $f(x, y, z) = (x+y)e^{x+2y+3z}$

Answer:  $\frac{\partial f}{\partial x} = e^{x+2y+3z} + (x+y)e^{x+2y+3z}$

$$\frac{\partial f}{\partial y} = e^{x+2y+3z} + 2(x+y)e^{x+2y+3z}$$

$$\frac{\partial f}{\partial z} = 3(x+y)e^{x+2y+3z}$$

Practise:  $f(x, y, z) = (2+3x+y)^z$

Recall  $(ax)' = ax^a \ln a$

$$\frac{\partial f}{\partial x} = 3z \cdot (2+3x+y)^{z-1}$$

$$(x^a)' = ax^{a-1}$$

$$\frac{\partial f}{\partial y} = z \cdot (2+3x+y)^{z-1}$$

$$\frac{\partial f}{\partial z} = (2+3x+y)^z \cdot \ln(2+3x+y)$$

Practise:  $f(x, y, z) = xye^{xy}$

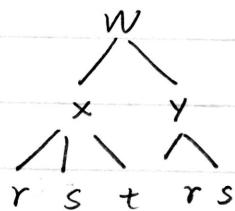
$$\frac{\partial f}{\partial x} = ye^{xy} + xy^2e^{xy}$$

$$\frac{\partial f}{\partial y} = xe^{xy} + x^2ye^{xy}$$

Examples:

$$\text{P994 #51 } W = \sqrt{x^2+y^2} \quad x = r^2 s t \quad y = r+s^2$$

Answer:



$$\frac{\partial W}{\partial x} = \frac{1}{2} (x^2+y^2)^{-\frac{1}{2}} \cdot 2x$$

$$\frac{\partial W}{\partial y} = \frac{1}{2} (x^2+y^2)^{-\frac{1}{2}} \cdot 2y$$

$$\frac{\partial x}{\partial r} = 2rs \cdot t \quad \frac{\partial y}{\partial s} = r^2 \quad \frac{\partial x}{\partial t} = r^2 s$$

$$\frac{\partial y}{\partial r} = 1 \quad \frac{\partial y}{\partial s} = 2s$$

$$\frac{\partial W}{\partial r} = \frac{\partial W}{\partial x} \frac{\partial x}{\partial r} = x(x^2+y^2)^{-\frac{1}{2}} \cdot 2r \cdot s \cdot t$$

$$\frac{\partial W}{\partial s} = \frac{\partial W}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial W}{\partial y} \frac{\partial y}{\partial s} = x(x^2+y^2)^{-\frac{1}{2}} \cdot r^2 + y(x^2+y^2)^{-\frac{1}{2}} \cdot 2s$$

$$\frac{\partial W}{\partial t} = \frac{\partial W}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial W}{\partial y} \frac{\partial y}{\partial t} = x(x^2+y^2)^{-\frac{1}{2}} \cdot r^2 s + y(x^2+y^2)^{-\frac{1}{2}} \cdot 1$$

Topic 2: Limit  $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y)$

Example: P993 #34

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin xy}{x^2+y^2}$$

$$\text{Answer: Let } x=y \quad \lim_{x \rightarrow 0} \frac{\sin x^2}{2x^2} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin x^2}{x^2} = \frac{1}{2}$$

$$\text{Let } y=2x \quad \lim_{x \rightarrow 0} \frac{\sin 2x^2}{3x^2} = \frac{2}{3} \lim_{x \rightarrow 0} \frac{\sin (2x^2)}{(2x^2)} = \frac{2}{3}$$

The limit DNE by two path test.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

Answer: Let  $u=x^2+y^2$  or use polar coordinates

$$\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1 \quad \#$$

Topic 3: Equation of planes and lines

Recall: Equation of line through  $(x_0, y_0, z_0)$  with direction  $\vec{v} = \langle a, b, c \rangle$  is

$$\vec{r}(t) = (x_0, y_0, z_0) + t \cdot \langle a, b, c \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

Equation of plane through  $(x_0, y_0, z_0)$  with normal vector  $\vec{n} = \langle a, b, c \rangle$  is

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

Example: Find the plane through  $(-1, 2, 3)$  and parallel to  $-x+y+3z=0$

Answer: normal vector  $\vec{n} = \langle -1, 1, 3 \rangle$

$$+1 \quad -2 \quad -9 \\ -1(x-1) + (y-2) + 3(z-3) = 0$$

$$-x+y+3z = 10$$

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P887 #36. The line through  $(0, 1, 4)$  and orthogonal to  $\langle -2, 1, 7 \rangle$  and  $y$ -axis

Answer:  $\vec{v}_1 = \langle -2, 1, 7 \rangle \quad \vec{v}_2 = \langle 0, 1, 0 \rangle$

$$\text{The direction } \vec{d} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ -2 & 1 & 7 \\ 0 & 1 & 0 \end{vmatrix} = -7\vec{i} - 0\vec{j} - 2\vec{k} \\ = \langle -7, 0, -2 \rangle$$

$$\text{Equation of line } \vec{r}(t) = (0, 1, 4) + t \cdot \langle -7, 0, -2 \rangle$$

$$= \langle 0 - 7t, 1 + 0t, 4 - 2t \rangle$$

$$= \langle -7t, 1, 4 - 2t \rangle$$

Practice: Equation of line of intersection of two planes P992 #5

Review Rec Notes on Oct. 3rd.

#### Topic 4 Geometric concepts

$$\textcircled{1} \text{ Angle between } \vec{u} \text{ and } \vec{v} \quad \cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}$$

$$\textcircled{2} \text{ Projection of } \vec{u} \text{ on } \vec{v} \quad \text{Proj}_{\vec{v}} \vec{u} = \underbrace{\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} * \frac{\vec{v}}{|\vec{v}|}}_{\text{Scalar } \vec{v}} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} * \vec{v}$$

$$\textcircled{3} \text{ Given } \vec{r}(t). \quad (\text{Table on Page 883})$$

$$\text{Practice} \quad \text{a. Unit tangent vector } \vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$\text{P889 #63} \quad \text{b. Principle unit normal vector } \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = \frac{d\vec{T}/ds}{|d\vec{T}/ds|}$$

$$\#71 \quad \text{c. Unit binormal vector. } \vec{B} = \vec{T} \times \vec{N}$$

$$\text{d. curvature } K(t) = \frac{|d\vec{T}/dt|}{|\vec{r}'(t)|} = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$

$$= \left| \frac{d\vec{T}}{ds} \right|$$

$$\text{e. Torsion: } \tau = - \frac{d\vec{B}}{ds} \cdot \vec{N} = \frac{(\vec{v} \times \vec{a}) \cdot \vec{a}'}{|\vec{v} \times \vec{a}|^2}$$

$$\textcircled{4} \text{ Length of curves } L = \int_a^b |\vec{r}'(t)| dt \quad \text{or} \quad L = \int_a^b \sqrt{f'(a)^2 + (f'(a))^2} da$$

$$\text{Relation of arc length } s \text{ and parameter } t \quad s = \int_a^t |\vec{r}'(u)| du$$

Detailed  
Solutions

Examples: P867

$$\#26. \vec{u} = -3\hat{j} + 4\hat{k} \quad \vec{v} = -4\hat{i} + \hat{j} + 5\hat{k} \quad (\vec{u} = \langle 0, -3, 4 \rangle \quad \vec{v} = \langle -4, 1, 5 \rangle)$$

$$\textcircled{1} \quad \cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} = \frac{-3+20}{5 \cdot \sqrt{16+1+25}} = \frac{17}{5\sqrt{42}}$$

$$\textcircled{2} \quad \text{Scal } \vec{v} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \frac{17}{\sqrt{42}}$$

$$\text{Proj } \vec{v} \vec{u} = \text{Scal } \vec{v} \vec{u} \cdot \frac{\vec{v}}{|\vec{v}|} = \frac{17}{\sqrt{42}} \cdot \frac{\langle -4, 1, 5 \rangle}{\sqrt{42}} = \frac{17}{42} \langle -4, 1, 5 \rangle$$

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#51 #57

$$\vec{r}(t) = \langle t^2, \frac{4\sqrt{2}}{3}t^{\frac{3}{2}}, 2t \rangle \quad 1 \leq t \leq 3$$

\textcircled{1} Arc length from  $t=1$  to  $t=3$

$$\vec{r}'(t) = \langle 2t, \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}t^{\frac{1}{2}}, 2 \rangle = \langle 2t, 2\sqrt{2}t^{\frac{1}{2}}, 2 \rangle$$

$$|\vec{r}'(t)| = \sqrt{4t^2 + 8t + 4} = \sqrt{4(t^2 + 2t + 1)} = 2(t+1)$$

$$L = \int_1^3 |\vec{r}'(t)| dt = \int_1^3 2(t+1) dt = t^2 + 2t \Big|_1^3 = 9 + 6 - 1 - 2 = 12$$

\textcircled{2} Find the description of curve that uses arc length as a parameter

Relation of  $s$  and  $t$  is

$$s = \int_1^t |\vec{r}'(u)| du = \int_1^t 2u+2 du = u^2 + 2u \Big|_1^t = t^2 + 2t - 3 = (t+1)^2 - 4$$

$$s+4 = (t+1)^2$$

$$t = \sqrt{s+4} - 1$$

$$\vec{r}(s) = \langle (\sqrt{s+4}-1)^2, \frac{4\sqrt{2}}{3}(\sqrt{s+4}-1)^{\frac{3}{2}}, 2(\sqrt{s+4}-1) \rangle \text{ and } 0 \leq s \leq 12$$

Topic 5. Parametric curves and Polar coordinates

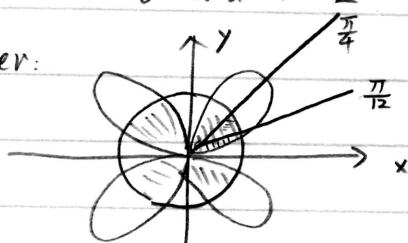
$$\textcircled{1} \text{ Derivative } x = x(t), y = y(t) \quad \frac{dy}{dx} = \frac{y'(t)}{x'(t)}, \quad r = f(\theta) \quad \frac{dy}{dx} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}$$

$$\textcircled{2} \text{ Area of regions in polar} \quad \text{Area} = \frac{1}{2} \int_a^b [f(\theta)^2 - g(\theta)^2] d\theta$$

Example: P771 #35

$$r = 4 \sin 2\theta \text{ and } r = 2$$

Answer:



2 intersections  $4 \sin 2\theta = 2$

$$\sin 2\theta = \frac{1}{2} \quad 2\theta = \frac{\pi}{6} \quad \theta = \frac{\pi}{12}$$

$$\text{Area} = 8 \cdot \left[ \frac{1}{2} \int_{0}^{\frac{\pi}{2}} (4\sin 2\theta)^2 d\theta + \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} (2)^2 d\theta \right]$$

$$\begin{aligned} &= 4 \left[ 16 \int_0^{\frac{\pi}{2}} \sin^2 2\theta d\theta + 4 \cdot \left( \frac{\pi}{4} - \frac{\pi}{2} \right) \right] \\ &= 4 \left[ 16 \cdot \int_0^{\frac{\pi}{2}} \frac{1 - \cos 4\theta}{2} d\theta + \frac{2\pi}{3} \right] \\ &= 4 \left[ 16 \cdot \left( \frac{1}{2}\theta - \frac{1}{8}\sin 4\theta \right) \Big|_0^{\frac{\pi}{2}} + \frac{2\pi}{3} \right] \\ &= 4 \left[ 16 \left( \frac{1}{2} \times \frac{\pi}{2} - \frac{1}{8} \sin \pi \right) + \frac{2\pi}{3} \right] \\ &= 4 \left[ \frac{4\pi}{3} - \sqrt{3} \right] \end{aligned}$$

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Example P767 #29

$$r = 4 + 2\sin\theta$$

- ① Find all points that the curve has vertical and horizontal tangent lines.

$$\text{Answer: } \frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

$$\text{a. Horizontal} \Leftrightarrow \frac{dy}{dx} = 0 \Leftrightarrow f'(\theta)\sin\theta + f(\theta)\cos\theta = 0 \text{ and } f'(\theta)\cos\theta - f(\theta)\sin\theta \neq 0$$

$$\begin{aligned} f'(\theta)\sin\theta + f(\theta)\cos\theta &= (2\cos\theta\sin\theta + 4\sin\theta\cos\theta + 2\sin\theta\cos\theta) \\ &= 4\cos\theta(\sin\theta + 1) = 0 \end{aligned}$$

$$\cos\theta = 0 \text{ or } \sin\theta = -1$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad \theta = \frac{3\pi}{2}$$

$$\begin{aligned} f'(\theta)\cos\theta - f(\theta)\sin\theta &= 2\cos\theta\cos\theta - 4\sin\theta - 2\sin^2\theta \\ &= 1 - 2\sin^2\theta - 4\sin\theta - 2\sin^2\theta \\ &= 1 - 4\sin^2\theta - 4\sin\theta \end{aligned}$$

$$\text{when } \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \quad f'(\theta)\cos\theta - f(\theta)\sin\theta \neq 0$$

Two points with horizontal tangent lines.  $(6, \frac{\pi}{2})$   $(2, \frac{3\pi}{2})$

$$\text{b. Vertical tangent line} \Leftrightarrow f'(\theta)\sin\theta + f(\theta)\cos\theta \neq 0 \text{ and } f'(\theta)\cos\theta + f(\theta)\sin\theta = 0$$

- ② Slope of tangent line at origin

$$r = 0 \Leftrightarrow 4 + 2\sin\theta = 0 \Leftrightarrow \sin\theta = -2, \text{ No solution}$$

The curve does not pass the origin