

§ 12.9

Quiz § 12.5-12.9

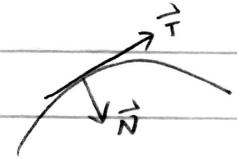
Definitions and formulas (PSS3 summary)

12.8 12.9

① Unit tangent vector  $\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$

② Curvature: the rate of change of  $\vec{T}$  with respect to arc length  $s$

$$k(t) = \frac{\|\vec{r}' \times \vec{r}''\|}{\|\vec{r}'\|^3} = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|^3} = \frac{\|d\vec{v}/dt\|}{\|\vec{r}'(t)\|}$$



③ Principle: Unit normal vector:  $\vec{N}(t) = \frac{d\vec{T}/dt}{\|d\vec{T}/dt\|}$

$\vec{N}(t) \perp \vec{T}(t)$  and points to the inside of the curve

④ Decompose acceleration  $\vec{a}$  into  $\vec{N}$  and  $\vec{T}$  direction, i.e.

$$\vec{a} = a_T \cdot \vec{T} + a_N \cdot \vec{N} \text{ where } a_T = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|} \quad a_N = \frac{\vec{v} \cdot \vec{a}}{\|\vec{v}\|} \quad \text{HW 6 # 49}$$

⑤ Unit binormal vector:  $\vec{B} = \vec{T} \times \vec{N}$

⑥ Torsion:  $\tau = -\frac{d\vec{B}}{dt} \cdot \vec{N} / \|\vec{r}'(t)\|$   $|\tau|$  is the rate that curve twists out the plane generated by  $\vec{T}$  and  $\vec{N}$

Examples: HW 5 # 44

Remark: if the curve is on a plane,  $\tau = 0$

$$\vec{r}(t) = \langle 5\cos t, 5\sin t, \sqrt{3}t \rangle$$

① Find  $\vec{T}(t)$  and  $\vec{N}(t)$

Answer:  $\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\langle -5\sin t, 5\cos t, \sqrt{3} \rangle}{\sqrt{25+3}} = \frac{\langle -5\sin t, 5\cos t, \sqrt{3} \rangle}{\sqrt{28}}$

$$\vec{T}'(t) = \frac{1}{\sqrt{28}} \langle -5\cos t, -5\sin t, 0 \rangle$$

$$\|\vec{T}'(t)\| = \frac{1}{\sqrt{28}} \cdot \sqrt{25\cos^2 t + 25\sin^2 t} = \frac{1}{\sqrt{28}} \sqrt{25} = \frac{5}{\sqrt{28}}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \langle -\cos t, \sin t, 0 \rangle$$

② Find curvature  $k(t)$

Answer:  $k = \frac{\|d\vec{T}/dt\|}{\|\vec{r}'(t)\|} = \frac{\frac{5}{\sqrt{28}}}{\sqrt{28}} = \frac{5}{28} \quad \#$

Note: You may also want to use  $k(t) = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|^3}$

Practise: Midterm problem

The acceleration of a particle from  $t=0$  to  $t=2\pi$  is given by the following function

$$\vec{a}(t) = \langle -3\sin t, -5\cos t, -4\sin t \rangle$$

③ The initial  $\vec{v}(0) = \langle 7, 0, 1 \rangle$ . Find  $\vec{v}(t)$  for  $0 \leq t \leq 2\pi$

Answer:  $\vec{v}(t) = \langle 31\cos t + 4, -5\sin t, 4\cos t - 3 \rangle$

(6) Find curvature of position function for  $0 \leq t \leq 2\pi$

Answer:  $\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$

Or you can use the formula

$$\kappa(t) = \frac{|d\vec{T}/dt|}{|\vec{r}'(t)|}$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} i & j & k \\ 3\cos t + 4 & -5\sin t & 4\cos t - 3 \\ -3\sin t & -5\cos t & -4\sin t \end{vmatrix}$$

$$= (20 - 15\cos t)i + (25\sin t)j + (15 + 20\cos t)k$$

$$|\vec{v}| = \sqrt{(3\cos t + 4)^2 + (-5\sin t)^2 + (4\cos t - 3)^2} = \sqrt{50}$$

$$|\vec{v} \times \vec{a}| = \sqrt{(20 - 15\cos t)^2 + (25\sin t)^2 + (15 + 20\cos t)^2} = 1025$$

$$\kappa = \frac{1025}{50\sqrt{50}}$$

#

Old Midterm problem

$$\vec{r} = \langle \cos t, \sin t, t \rangle. \text{ Find torsion for the helix using } \tau = \frac{(\vec{r}' \times \vec{r}'') \cdot \vec{r}'''}{|\vec{r}' \times \vec{r}''|^2}$$

$$\text{Answer: } \tau = \frac{1}{2}$$

#

Old Midterm Problem

Let  $\vec{r}(t) = \langle t^2, 1, t \rangle$ . For what value this curve have max curvature?

Answer:  $\vec{v} = \vec{r}'(t) = \langle 2t, 0, 1 \rangle$

$$\vec{a} = \vec{v}'(t) = \langle 2, 0, 0 \rangle$$

Or you can use the formula

$$\vec{v} \times \vec{a} = \begin{vmatrix} i & j & k \\ 2t & 0 & 1 \\ 2 & 0 & 0 \end{vmatrix} = \langle 0, 2, 0 \rangle$$

$$\kappa(t) = \frac{|d\vec{T}/dt|}{|\vec{r}'(t)|}$$

$$|\vec{v} \times \vec{a}| = 2$$

$$|\vec{v}| = \sqrt{4t^2 + 1}$$

$$\kappa(t) = \frac{2}{(4t^2 + 1)^{\frac{3}{2}}}$$

$\kappa(t)$  researches its max when  $t = 0$

#

Old Midterm Problem

$$\vec{r}(t) = \langle \cos(t^2), \sin(t^2) \rangle \quad 0 \leq t \leq 2\pi$$

① Find  $\vec{T}(t)$  ② Find  $\vec{N}(t)$

③ What are  $\vec{T}$  and  $\vec{N}$  at the points of the curve corresponding to  $t = 0$  and  $t = \sqrt{\frac{\pi}{2}}$ ?

Write the vectors and make a sketch of C with those vectors.

## § 13.1 - 13.2

### Topic 1: Equation of planes

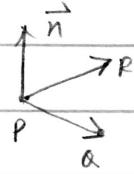
Given a point  $P_0(x_0, y_0, z_0)$  on the plane and the normal vector  $\vec{n} = \langle a, b, c \rangle$ , the equation of the plane is  $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

Examples: Old Midterm

Let  $P=(1, 1, 1)$   $Q=(0, 3, 1)$  and  $R=(0, 1, 4)$

① Find equation of plane that passes through  $P, Q, R$

Answer:



$$\text{normal vector} = \vec{PQ} \times \vec{PR}$$

$$\vec{PQ} = \langle -1, 2, 0 \rangle$$

$$\vec{PR} = \langle -1, 0, 3 \rangle$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \langle 6, 3, 2 \rangle = \vec{n}$$

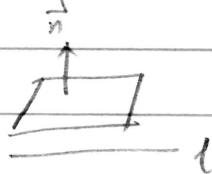
Equation of plane

$$6(x-1) + 3(y-1) + 2(z-1) = 0$$

$$\Leftrightarrow 6x + 3y + 2z = 11 \quad \#$$

② Is the line through  $(1, 2, 3)$  and  $(2, 2, 0)$  parallel to the plane?

Answer



$l \parallel \text{plane } PQR \Leftrightarrow \vec{v} \perp \vec{n}$ , where  $\vec{v}$  is the direction of line

$$\vec{v} = \langle 2-1, 2-2, 0-3 \rangle = \langle 1, 0, -3 \rangle$$

$$\vec{v} \cdot \vec{n} = \langle 1, 0, -3 \rangle \cdot \langle 6, 3, 2 \rangle = 6-6=0 \quad \text{Yes! } \vec{v} \perp \vec{n} \quad \#$$

Old Midterm :

Consider the following planes

$$x-3y=0$$

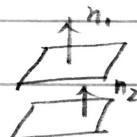
$$x+y+z=-2$$

a) show that these planes are not parallel?

Answer:  $\vec{n}_1 = \langle 1, -3, 0 \rangle$   $\vec{n}_2 = \langle 1, 1, 1 \rangle$

If two planes are parallel,  $\vec{n}_1 \parallel \vec{n}_2$

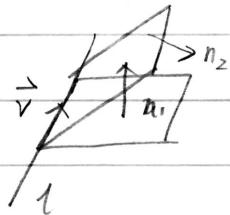
$$\vec{n}_1 \parallel \vec{n}_2 \Leftrightarrow \text{there exists } \lambda \neq 0 \text{ such that } \vec{n}_1 = \lambda \vec{n}_2$$



$$\Leftrightarrow \begin{cases} 1 = \lambda \cdot 1 \\ -3 = \lambda \cdot 1 \\ 0 = \lambda \cdot 1 \end{cases} \quad \text{No solution} \Rightarrow \lambda \text{ does not exist. } \vec{n}_1 \times \vec{n}_2$$

⑥ Find the equation for the line of intersection of the two planes

Answer:



The direction of  $\ell \perp$  both  $\vec{n}_1$  and  $\vec{n}_2$

$$\text{So } \vec{v} = \vec{n}_1 \times \vec{n}_2$$

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 0 & -3 \\ 1 & 1 & 1 \end{vmatrix} = \langle 3, -4, 1 \rangle$$

How do we find a point on the line?

$$\begin{cases} x - 3z = 0 & \text{solve for intersections} \\ x + y + z = -2 \end{cases}$$

Since we only need one point, we can choose special point

$$\text{Let } y = 0 \quad \begin{cases} x - 3z = 0 \\ x + z = -2 \end{cases} \Rightarrow \begin{cases} x = -\frac{3}{2} \\ z = -\frac{1}{2} \end{cases} \quad \left( -\frac{3}{2}, 0, -\frac{1}{2} \right)$$

So the equation of line is

$$\begin{aligned} \vec{r}(t) &= \left( -\frac{3}{2}, 0, -\frac{1}{2} \right) + t \cdot \langle 3, -4, 1 \rangle \\ &= \left\langle 3t - \frac{3}{2}, -4t, t - \frac{1}{2} \right\rangle \end{aligned} \quad \#$$

Other definitions

① Quadratic Surface: in general  $F(x, y, z) = 0$

HW7 #4 Special case: cylinder: only two variables, e.g.  $x^2 - y = 0$  is a cylinder // z-axis

HW7 #5 { ② x-axis intersections: let  $y = z = 0$  and solve for x

#6 ③ xy trace: let  $z = 0$  and find the equation in terms of x and y

④ Functions with two variables, in general  $z = f(x, y)$

HW7 #11-13 Domain of  $z = f(x, y)$  is a region in  $R^2$

$$\text{e.g.: } f(x, y) = \sqrt{27 - x^2 - y^2} \quad \text{Domain: } \{(x, y) \mid x^2 + y^2 \leq 27\}$$

(5) Level curve

Given  $z = f(x, y)$ , the level curve for level  $z_0$  is  $z_0 = f(x, y)$

HW7 #14 -16

