

Nov. 7

§13.6 - 14.6 Exam 3 Review

[Topic 1] Directional Derivative and Gradient

Gradient of f at (x, y) is $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$ (vector)

Directional derivative of f at (a, b) in the direction of \vec{u} is $\nabla f(a, b) \cdot \frac{\vec{u}}{|\vec{u}|}$ (scalar)

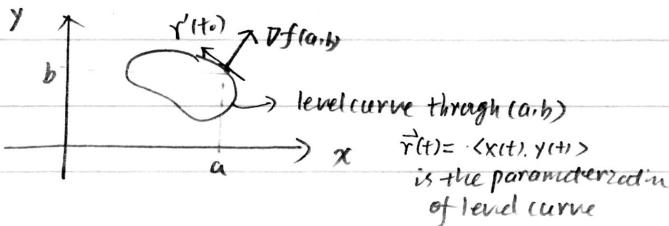
Results:

① f has max rate of increase at (a, b) in the direction of $\nabla f(a, b)$, and the rate of change is $|\nabla f(a, b)|$

② f has min. decrease $-\nabla f(a, b)$

③ f has no change at (a, b) in the direction that is orthogonal to $\nabla f(a, b)$

④ The tangent line of level curve of f passing (a, b) is orthogonal to $\nabla f(a, b)$



Remark

$$\langle a, b \rangle \perp \langle b, -a \rangle$$

$$\langle a, b \rangle \perp \langle -b, a \rangle$$

Example Pg 94 # 67

$$f(x, y) = \sqrt{4-x^2-y^2} = P(-1, 1)$$

$$\text{Answer: } \nabla f = \langle f_x, f_y \rangle = \left\langle \frac{-x}{\sqrt{4-x^2-y^2}}, \frac{-y}{\sqrt{4-x^2-y^2}} \right\rangle$$

$$\text{Steepest ascent: } \nabla f(-1, 1) = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

$$\text{Steepest descent: } -\nabla f(-1, 1) = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$\text{No change: } \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \text{ or } \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

$$\text{Directional derivative in } \vec{u} = \langle 2, 1 \rangle = \nabla f(-1, 1) \cdot \frac{\vec{u}}{|\vec{u}|} = \frac{\left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle \cdot \langle 2, 1 \rangle}{\sqrt{5}} = \frac{1}{\sqrt{10}}$$

[Topic 2] Tangent Plane and Linear Approximation

- Tangent plane of $F(x, y, z) = 0$ at $P_0(a, b, c)$ is the plane passing through P_0 with normal vector $\nabla F(a, b, c)$. The equation is

$$F_x(a, b, c)(x-a) + F_y(a, b, c)(y-b) + F_z(a, b, c)(z-c) = 0$$

$$\text{Special case: } z = f(x, y) \Leftrightarrow \underline{f(x, y) - z = 0} \quad \nabla F = \langle f_x, f_y, -1 \rangle$$

- The linear approximation to surface $z = f(x, y)$ at $(a, b, f(a, b))$ is the plane tangent plane at the point, given by

$$z = L(x, y) = f_x(a, b)(x-a) + f_y(a, b)(y-b) + f(a, b)$$

The differential $\Delta z \approx f_x(x,y)\Delta x + f_y(x,y)\Delta y$

P995 #81 *

P995 Review #77

$$z = \ln(1+xy) \quad (1,2, \ln 3) \text{ tangent plane}$$

$$\text{Answer: } F(x,y,z) = \ln(1+xy) - z$$

$$\nabla F = \langle F_x, F_y, F_z \rangle = \left\langle \frac{y}{1+xy}, \frac{x}{1+xy}, -1 \right\rangle$$

$$\nabla F(1,2,\ln 3) = \left\langle \frac{2}{3}, \frac{1}{3}, -1 \right\rangle$$

$$\text{Tangent Plane: } \frac{2}{3}(x-1) + \frac{1}{3}(y-2) - (z-\ln 3) = 0$$

$$z = \frac{2}{3}(x-1) + \frac{1}{3}(y-2) + \ln 3 \quad (\text{linear approximation at } (1,2,\ln 3))$$

Topic 3: Classification of critical points

2nd derivative test:

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2$$

① $D(a,b) > 0 \quad f_{xx}(a,b) < 0 \quad (a,b) \rightarrow \text{local max}$

② $D(a,b) > 0 \quad f_{xx}(a,b) > 0 \quad (a,b) \rightarrow \text{local min}$

③ $D(a,b) < 0 \quad (a,b) \rightarrow \text{saddle}$

④ $D(a,b) = 0, \text{ inconclusive}$

Recall: critical points $\begin{cases} f_x = 0 \text{ solution} \\ f_y = 0 \end{cases}$

Example: P995 #87

Topic 4: Absolute max/min & Max/min subject to constraint.

To find absolute max/min

Step 1: Evaluate f at critical points in $R \quad \rightarrow \text{compare and choose max/min}$

Step 2: Find max/min of f on the boundary of R

Method: parameterization, Lagrange multiplier

Lagrange multiplier: find max/min of f subject to constraint $g=0$

Setup $\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$ and solve for x, y

Evaluate f at the solutions

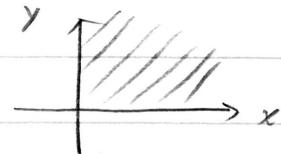
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Example P983 # 55

$$f(x,y) = 2e^{-x-y} \quad R = \{(x,y) : x > 0, y > 0\}$$

Answer: step 1. Evaluate f at critical pts

$$\begin{cases} f_x = -2e^{-x-y} = 0 \\ f_y = -2e^{-x-y} = 0 \end{cases} \quad \text{No solution!}$$



Step 2: Evaluate f on the boundary of R

$$\text{Boundary: } x=0 \quad y \geq 0 \quad f(y) = 2e^{-y} \quad f_{\max} = 2 \text{ when } y=0 \\ f_{\min} = 0 \text{ when } y \rightarrow \infty$$

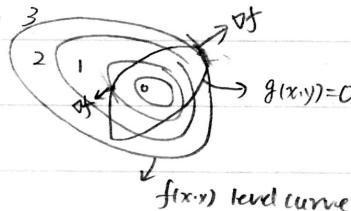
$$y=0 \quad x > 0 \quad f(x) = 2e^{-x} \quad f_{\max} = 2 \text{ when } x=0$$

$$f_{\min} = 0 \text{ when } x \rightarrow \infty$$

f has absolute max 2, no absolute min value

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* Example P991 # 51 Important!



$$f_{\max} = 3 \text{ subject to } g=0$$

$$f_{\min} = 1 \text{ subject to } g=0$$

$\nabla f \perp$ tangent line of level curve

Example. (Final)

$$R = \{(x,y) : 16x^2 + 16y^2 \leq 1\} \quad f(x,y) = x^2 - y^2 + x$$

Find abs max/min of f on R

Answer: step 1: Evaluate f at critical points in R

$$\begin{cases} f_x = 2x+1 = 0 \\ f_y = -2y = 0 \end{cases} \quad (-\frac{1}{2}, 0) \notin R !$$

Step 2: Evaluate f on the boundary $x^2 + y^2 = \frac{1}{16}$

Method I: Substitution: $y^2 = \frac{1}{16} - x^2$

$$f(x) = x^2 - (\frac{1}{16} - x^2) + x = 2x^2 + x - \frac{1}{16} = 2(x + \frac{1}{4})^2 - \frac{3}{16}$$

$$f_{\max} = f(\frac{1}{4}) = \frac{1}{2} - \frac{3}{16} = \frac{1}{16}$$

$$f_{\min} = f(-\frac{1}{4}) = -\frac{3}{16}$$

3

Method II Parameterization: $x = \frac{1}{4} \cos t$ $y = \frac{1}{4} \sin t$

$$\begin{aligned} f(t) &= (\frac{1}{4} \cos t)^2 - (\frac{1}{4} \sin t)^2 + \frac{1}{4} \cos t \\ &= \frac{1}{16} \cos^2 t - \frac{1}{16} \sin^2 t + \frac{1}{4} \cos t \\ &= \frac{1}{16} \cos^2 t - \frac{1}{16} (1 - \cos^2 t) + \frac{1}{4} \cos t \\ &= \frac{1}{8} \cos^2 t + \frac{1}{4} \cos t - \frac{1}{16} \\ &= \frac{1}{8} (\cos t + 1)^2 - \frac{3}{16} \end{aligned}$$

$$f_{\max} = \frac{1}{8} \times 2^2 - \frac{3}{16} = \frac{1}{2} - \frac{3}{16} = \frac{1}{16} \quad \text{when } \cos t = 1 \text{ and } x = \frac{1}{4}, y = 0$$

$$f_{\min} = \frac{1}{8} \times 0 - \frac{3}{16} = -\frac{3}{16} \quad \text{when } \cos t = -1 \text{ and } x = -\frac{1}{4}, y = 0$$

Method III: Lagrange multiplier

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \Leftrightarrow \begin{cases} 2x+1 = \lambda \cdot 2x & (1) \\ -2y = \lambda \cdot 2y & (2) \\ x^2+y^2 = \frac{1}{16} & (3) \end{cases}$$

$$(2) \Rightarrow y = 0 \text{ or } \lambda = -1$$

$$\text{when } y = 0 \cdot (3) \Rightarrow x = \pm \frac{1}{4} \quad (\frac{1}{4}, 0), (-\frac{1}{4}, 0)$$

$$\text{when } \lambda = -1 \cdot (1) \Rightarrow x = -\frac{1}{4} \quad (-\frac{1}{4}, 0)$$

$$\text{Step 3: } f_{\max} = \frac{1}{16} \text{ and } f_{\min} = -\frac{3}{16} \quad \#$$

Practice P 995 # 91

Example 99.

Find the points on the cone $z^2 - x^2 - y^2 = 0$ that are closest to the point $(1, 3, 1)$

Answer: Objective function $f(x, y, z) = d^2 = (x-1)^2 + (y-3)^2 + (z-1)^2$

Constraint: $z^2 - x^2 - y^2 = 0$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \Leftrightarrow \begin{cases} 2(x-1) = -2x\lambda & (1) \\ 2(y-3) = -2y\lambda & (2) \\ 2(z-1) = 2z\lambda & (3) \\ z^2 - x^2 - y^2 = 0 & (4) \end{cases} \Rightarrow \begin{aligned} x &= \frac{1}{1+\lambda} \\ y &= \frac{3}{1+\lambda} \\ z &= \frac{1-\lambda}{1+\lambda} \end{aligned} \quad \text{plug in (4)}$$

$$\text{Solution } (0, 0, 0) \quad (\frac{1}{2} \pm \frac{\sqrt{10}}{20}, \frac{3}{2} \pm \frac{3\sqrt{10}}{20}, \frac{1}{2} \pm \frac{\sqrt{10}}{2})$$

$$f_{\min} = \frac{11}{2} - \sqrt{10}$$

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Topic 5 Double integral.

Application: Area of $R = \iint_R 1 dA$

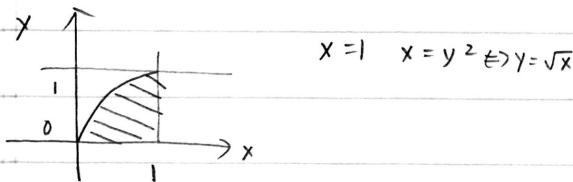
Average value of f : $\bar{f} = \frac{\iint_R f dA}{\iint_R 1 dA}$

Volume of solid = $\iint_R f_1 - f_2 dA$. R : projection of solid

Two types of double integral: cartesian $dA = dx dy = dy dx$

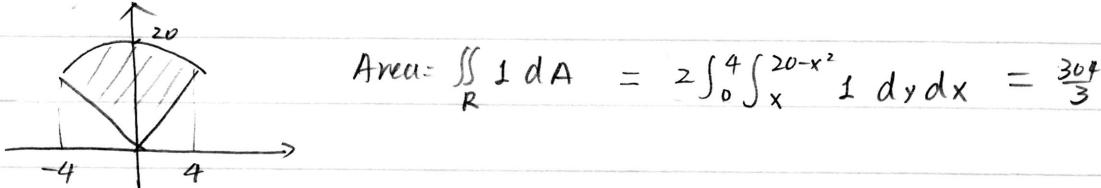
Polar $dA = r dr d\theta$

Example: $\int_0^1 \int_{y^2}^1 4ye^{x^2} dx dy$

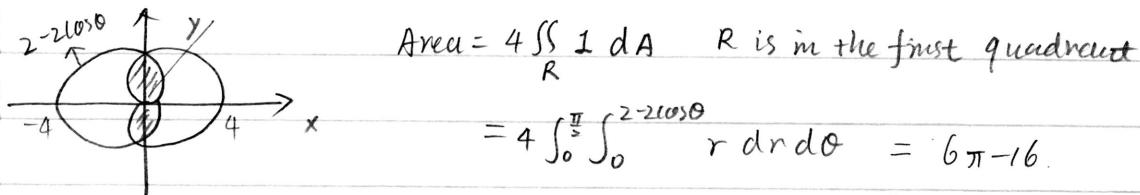


$$\int_0^1 \int_{y^2}^{\sqrt{x}} 4ye^{x^2} dy dx = \int_0^1 2y^2 e^{x^2} \Big|_{y^2}^{\sqrt{x}} dx = \int_0^1 2x e^{x^2} dx = e^{x^2} \Big|_0^1 = e - 1$$

P1080 #9 Area of R bdd by $y = |x|$ and $y = 20 - x^2$



#21. Area of region lies inside both $r = 2 - 2 \cos \theta$ and $r = 2 + 2 \cos \theta$



(need double angle formula)

Topic 6 Triple integral

Application: ① Volume = $\iiint_D 1 dv$

② $\bar{f} = \frac{\iiint_D f dv}{\iiint_D 1 dv}$

③ Mass = $\iiint_D \rho dv$

④ Moment $M_{xy} = \iiint_D z \rho dv$

⑤ Center. $\bar{z} = \frac{M_{xy}}{M}$

Three types triple integral

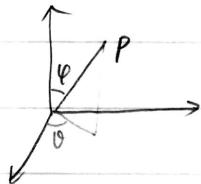
$$\textcircled{1} \text{ Cartesian } \iiint_D f dV = \iint_{R_{xy}} \int_{G(x,y)}^{H(x,y)} f dz dA = \iint_{R_{xz}} \int_{G(x,z)}^{H(x,z)} f dy dA$$

$$= \iint_{R_{yz}} \int_{G(y,z)}^{H(y,z)} f dx dA$$

$$\textcircled{2} \text{ Cylindrical. } \iiint_D f dV = \int_0^R \int_{r_1(0)}^{r_2(0)} \int_{G_1(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} f dz r dr d\theta$$

$\underbrace{R_{xy} \text{ polar}}$ $\underbrace{\text{Polar}}$

$$\textcircled{3} \text{ Spherical. } \iiint_D f dV = \int_0^{\rho_1} \int_0^{\rho_2} \int_{\rho_1}^{\rho_2} f(p \sin\varphi \cos\theta, p \sin\varphi \sin\theta, p \cos\varphi) p^2 \sin\varphi dp d\theta d\varphi$$



$$x = p \sin\varphi \cos\theta$$

$$y = p \sin\varphi \sin\theta$$

$$z = p \cos\varphi$$

$$0 \leq \varphi \leq \pi \quad 0 \leq \theta \leq 2\pi$$

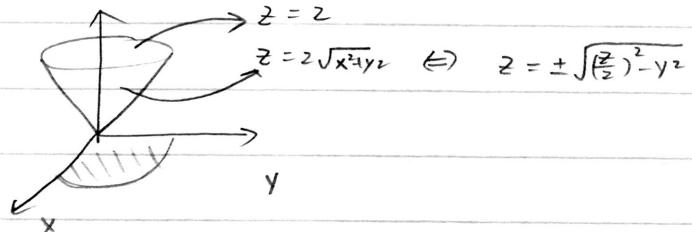
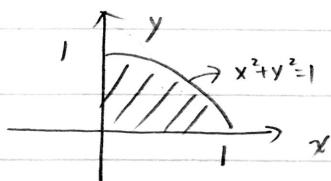
Please review the problems Z solved in previous recitation on spherical coordinates

Example: P108b

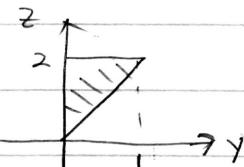
#25. Change order

$$\iint_D \int_0^{\sqrt{1-x^2}} \int_{2\sqrt{x^2+y^2}}^2 f dz dy dx \rightarrow dx dz dy$$

$\underbrace{R_{xy}}$



$$\iint_{R_{zy}} \int_{G(y,z)}^{H(y,z)} f dx dz dy$$

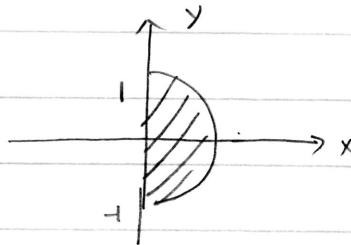


$$\int_0^1 \int_{2y}^2 \int_0^{\sqrt{z^2-y^2}} f dx dz dy$$

$$\#43 \int_{-1}^1 \int_{-2}^2 \int_0^{\sqrt{1-y^2}} \frac{1}{(1+x^2+y^2)^2} dx dz dy \rightarrow dz r dr d\theta$$

$$= \underbrace{\int_{-1}^1 \int_0^{\sqrt{1-y^2}}}_{R_{xy}} \int_{-2}^2 \frac{1}{(1+x^2+y^2)^2} dz dx dy$$

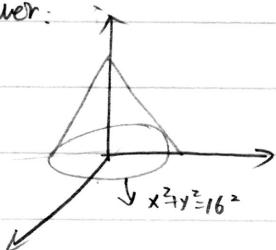
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \int_{-2}^2 \frac{1}{(1+r^2)^2} dz r dr d\theta$$



Final problem:

① Mass of solid bdd by $Z = 16 - \sqrt{x^2 + y^2}$, $Z = 0$ and $P = Z$

Answer:



$$M = \iiint_D p dv = \int_0^{2\pi} \int_0^{16} \int_0^{16-\sqrt{r^2}} z dz r dr d\theta$$

② Center of mass \bar{x} \bar{y} \bar{z}

Answer: Both the solid and $P = Z$ symmetric with respect to yz -plane xz -plane

$$\Rightarrow \bar{x} = \bar{y} = 0$$

To find \bar{z} .

$$M_{xy} = \iiint_D z p dv = \int_0^{2\pi} \int_0^{16} \int_0^{16-r} z^2 dz r dr d\theta$$

$$\bar{z} = \frac{M_{xy}}{M}$$

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P1082 # 57

(center of upper half ball) $\{(P, \varphi, \theta) : 0 \leq P \leq 16, 0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \theta \leq 2\pi\}$ with

$$P = 1 + \frac{P}{4}$$

Answer: $M = \iiint_D (1 + \frac{P}{4}) dv = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{16} (1 + \frac{P}{4}) P^2 \sin \varphi dP d\varphi d\theta$

$$M_{yz} = \iiint_D z (1 + \frac{P}{4}) dv = \iiint_D P \cos \varphi \cdot (1 + \frac{P}{4}) dv = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{16} (1 + \frac{P}{4}) P^3 \sin \varphi \cos \varphi dP d\varphi d\theta$$

$$\bar{z} = M_{yz}/M$$

$\bar{x} = \bar{y} = 0$, since both the solid and the density are symmetric