

§ 15.6 Surface Integral

Recall: single integral $\int_a^b f(x) dx$ integral along straight line $a \leq x \leq b$

double integral $\iint_R f dA$: integral on 2D region R

triple integral $\iiint_D f dV$: integral on 3D solid D .

△ line integral: $\int_C f ds$ or $\int_C \vec{F} \cdot d\vec{r}$ C is a curve in 2D or 3D space

△ Today: surface integral $\iint_S f ds$ or $\iint_S \vec{F} \cdot \vec{n} dS$: S is a

Type I: Surface integral of scalar valued function

Similar to $\int_C f ds$, we first need to parameterize the surface

Surface	Equation	Parametric form	$\vec{r}_u \times \vec{r}_v$	$ \vec{r}_u \times \vec{r}_v $
cylinder	$x^2 + y^2 = a^2 \quad 0 \leq z \leq h$	$\vec{r} = \langle a \cos u, a \sin u, v \rangle \quad 0 \leq u \leq 2\pi, 0 \leq v \leq h$	$\vec{r}_u \times \vec{r}_v$	Textbook
sphere	$x^2 + y^2 + z^2 = a^2$	$\vec{r} = \langle a \sin u \cos v, a \sin u \sin v, a \cos u \rangle \quad 0 \leq u \leq \pi, 0 \leq v \leq 2\pi$	1154	1154
Cone	$z^2 = x^2 + y^2 \quad 0 \leq z \leq h$	$\vec{r} = \langle v \cos u, v \sin u, v \rangle \quad 0 \leq u \leq 2\pi, 0 \leq v \leq h$		
✓ parabola	$z = x^2 + y^2 \quad 0 \leq z \leq h$	$\vec{r} = \langle v \cos u, v \sin u, v^2 \rangle \quad 0 \leq u \leq 2\pi, 0 \leq v \leq \sqrt{h}$		
✓ general surface	$z = g(x, y)$	$\vec{r} = \langle u, v, g(u, v) \rangle$		

$$\iint_S f(x, y, z) ds = \iint_R f(x(u, v), y(u, v), z(u, v)) \cdot |\vec{r}_u \times \vec{r}_v| dA \quad \text{where}$$

$R = \{(u, v) : a \leq u \leq b, c \leq v \leq d\}$ \vec{r}_u, \vec{r}_v are partial derivatives

Special case: If the surface is $z = g(x, y)$

$$\iint_S f(x, y, z) ds = \iint_R f(x, y, g(x, y)) \sqrt{1 + z_x^2 + z_y^2} dA$$

Note: $\vec{r} = \langle x, y, g(x, y) \rangle$

$$\vec{r}_x = \langle 1, 0, z_x \rangle \quad \vec{r}_y = \langle 0, 1, z_y \rangle$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} = -z_x i + z_y j + l$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{z_x^2 + z_y^2 + l^2}$$

Examples:

- Use a surface integral to calculate the surface area of the part of the hyperboloid $z = x^2 + y^2$ above the region R in xy -plane which is the quarter disk of radius 4

$$R = \{(x, y) : 0 \leq x \leq 4, 0 \leq y \leq \sqrt{16 - x^2}\}$$

Answer:

$$\text{Area} = \iint_S 1 \, ds = \iint_R 1 \cdot \sqrt{1+8x^2+8y^2} \, dA$$

$$z = x^2 - y^2 \quad z_x = 2x \quad z_y = -2y$$

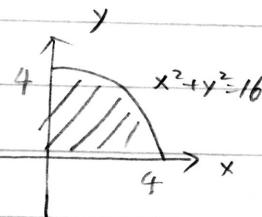
$$\text{Area} = \iint_R \sqrt{1+4x^2+4y^2} \, dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^4 \sqrt{1+r^2} \cdot r \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3} (1+r^2)^{\frac{3}{2}} \Big|_0^4 \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{3} (1+16)^{\frac{3}{2}} - \frac{1}{3} \right] \, d\theta$$

$$= \frac{\pi}{2} \cdot \left[\frac{1}{3} \cdot 17^{\frac{3}{2}} - \frac{1}{3} \right]$$



$$\int \sqrt{1+r^2} \cdot r \, dr$$

$$u = 1+r^2 \quad du = 2r \, dr$$

$$= \int \sqrt{u} \frac{1}{2} \, du$$

$$= \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}}$$

$$= \frac{1}{3} (1+r^2)^{\frac{3}{2}}$$

P1184 #45 Area of hemisphere $x^2+y^2+z^2=9 \quad z \geq 0$

$$\text{Answer: } \vec{r}(u,v) = \langle 3\sin u \cos v, 3\sin u \sin v, 3\cos u \rangle$$

$$0 \leq u \leq \frac{\pi}{2} \quad 0 \leq v \leq 2\pi$$

$$\vec{r}_u = \langle 3\cos u \cos v, 3\cos u \sin v, -3\sin u \rangle$$

$$\vec{r}_v = \langle -3\sin u \sin v, 3\sin u \cos v, 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 3\cos u \cos v & 3\cos u \sin v & -3\sin u \\ -3\sin u \sin v & 3\sin u \cos v & 0 \end{vmatrix} = \langle 9\sin^2 u \cos v, 9\sin^2 u \sin v, 9\sin u \cos u \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = 9\sin u$$

$$\text{Area} = \iint_S 1 \, ds = \iint_R |\vec{r}_u \times \vec{r}_v| \, dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 9\sin u \, dv \, du$$

$$= \int_0^{\frac{\pi}{2}} 2\pi 9\sin u \, du$$

$$= 18\pi \left(-\cos u \Big|_0^{\frac{\pi}{2}} \right)$$

$$= 18\pi$$

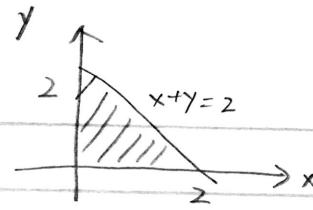
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HW 12 #49 $\iint_S (1+yz) \, ds$ S: $x+y+z=2$ first octant

#14, 15

Answer: $z = 2 - x - y$

$$zx = -1 \quad zy = -1$$



$$\begin{aligned} \iint_S (1+x+z) ds &= \iint_R 1+y(2-x-y) \sqrt{1+2x^2+2y^2} dA \\ &= \int_0^2 \int_{0-x}^{2-x} 1+y(2-x-y) \sqrt{3} dy dx \\ &= \frac{8\sqrt{3}}{3} \end{aligned}$$

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Type II: Surface integral of a vector field

$$\iint_S \vec{F} \cdot \vec{n} ds = \iint_R \vec{F}(u,v) \cdot (\pm \vec{r}_u \times \vec{r}_v) dA \quad " \pm " \text{ depends on } \vec{n}$$

Special: surface has explicit equation $z = g(x,y)$

$$\iint_S \vec{F} \cdot \vec{n} ds = \iint_R \vec{F}(x,y, z(x,y)) \cdot (\pm \langle -z_x, -z_y, 1 \rangle) dA \quad " \pm " \text{ depends on } \vec{n}$$

Example HW12 #48 (HW12 #19)

Find $\iint_S \vec{F} \cdot \vec{n} ds$ \vec{n} points in positive y-axis

$\vec{F} = \langle -y, x, 1 \rangle$ and S is cylinder $y = x^2$, $0 \leq x \leq 1$, $0 \leq z \leq 4$

Answer: $\vec{r}(u,v) = \langle u, u^2, v \rangle \quad 0 \leq u \leq 1 \quad 0 \leq v \leq 4$

$$\vec{r}_u = \langle 1, 2u, 0 \rangle \quad \vec{r}_v = \langle 0, 0, 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2u \vec{i} - \vec{j} + 0 \vec{k} \\ = \langle 2u, -1, 0 \rangle$$

$$\iint_R \langle -u^2, u, 1 \rangle \cdot (-\langle 2u, -1, 0 \rangle) dA$$

$$= \int_0^1 \int_0^4 (2u^3 + u) dv du$$

$$= 4$$

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HW12 #17 $\vec{F} = \langle e^{-y}, 5z, 2xy \rangle$ $S = \{(x,y,z) : z = \cos y \text{ for } |y| \leq \pi, 0 \leq x \leq 3\}$ upward

Answer: $\iint_S \vec{F} \cdot \vec{n} ds = \iint_R \vec{F} \cdot (\pm \langle -z_x, -z_y, 1 \rangle) dA \quad z_x = 0 \quad z_y = -\sin y$

$$= \iint_R \langle e^{-y}, 5\cos y, 2xy \rangle \cdot \langle 0, \sin y, 1 \rangle dA \quad \langle -z_x, -z_y, 1 \rangle = \langle 0, \sin y, 1 \rangle \text{ upward}$$

$$= \iint_R 5\sin y \cos y + 2xy dA = \int_0^3 \int_{-\pi}^{\pi} 5\sin y \cos y + 2xy dy dx$$

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Green's thm.

Conditions: Region R is 2D, connected, simply connected with a simple closed piecewise smooth counterclockwise oriented boundary C

$\vec{F} = \langle f, g \rangle$ and f and g have continuous first partial derivative on R

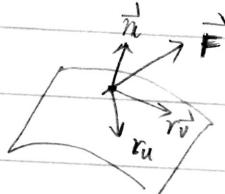
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R 2D \operatorname{curl} \vec{F} dA = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

$$\oint_C \vec{F} \cdot \vec{n} ds = \iint_R \nabla \cdot \vec{F} dA = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA$$

- Flux integral of vector field \vec{F} on oriented surface S

$$\iint_S \vec{F} \cdot \vec{n} ds \text{ or } \iint_S \vec{F} \cdot d\vec{s}$$

Assume that S has parametric form $\vec{r}(u, v)$. \vec{n} : unit normal vector



\vec{r}_u, \vec{r}_v provide two tangent directions

$$\vec{n} = \pm \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \quad " \pm " \text{ depends on the orientation of surface}$$

$$\text{Recall } ds \approx |\vec{r}_u \Delta u \times \vec{r}_v \Delta v| \approx |\vec{r}_u \times \vec{r}_v| |dudv|$$

$$\iint_S \vec{F} \cdot \vec{n} ds = \iint_R \vec{F}(u, v) \cdot \frac{(\pm \vec{r}_u \times \vec{r}_v)}{|\vec{r}_u \times \vec{r}_v|} \cdot |\vec{r}_u \times \vec{r}_v| dA$$

$$= \iint_R \vec{F}(u, v) (\pm \vec{r}_u \times \vec{r}_v) dA \quad R: \text{region in } u, v \text{ space}$$

Special: $z = z(x, y)$ explicit form

$$\vec{r}(x, y) = \langle x, y, z(x, y) \rangle$$

$$\vec{r}_x = \langle 1, 0, z_x \rangle \quad \vec{r}_y = \langle 0, 1, z_y \rangle$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} = -z_x \vec{i} - z_y \vec{j} + \vec{k} \\ = \langle -z_x, -z_y, 1 \rangle$$

$$\iint_S \vec{F} \cdot \vec{n} ds = \iint_R \vec{F}(x, y, z(x, y)) \cdot (\pm \langle -z_x, -z_y, 1 \rangle) dA. \quad R: \text{region in } xy \text{ space}$$