

§ 15.4 - 15.5

Topic 1: Divergence and curl (§ 15.5)

Assume $\vec{F} = \langle f, g, h \rangle$.

Def. $\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$. measure of expansion.

If $\nabla \cdot \vec{F} = 0$, then \vec{F} is called source free.

Special case. $\vec{F} = \langle f, g \rangle$. $\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$

Def. $\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \vec{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \vec{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \vec{k}$
measure of rotation.

If $\nabla \times \vec{F} = \vec{0}$, then \vec{F} is called irrotational.

Special case: $\vec{F} = \langle f, g \rangle$, 2D $\operatorname{curl} \vec{F} = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$.

Properties. (True or False, multiple-choice)

① Radial field $\vec{F} = \frac{\vec{r}}{|\vec{r}|^p} = \frac{\langle x, y, z \rangle}{(\sqrt{x^2+y^2+z^2})^p}$, $\nabla \cdot \vec{F} = \frac{3-p}{|\vec{r}|^{p+1}}$, $\nabla \times \vec{F} = \vec{0}$

② Rotation field $\vec{F} = \vec{a} \times \vec{r}$. $\vec{r} = \langle x, y, z \rangle$, $|\nabla \times \vec{F}| = 2|\vec{a}|$ and $\nabla \cdot \vec{F} = 0$

③ $\nabla \cdot (\nabla \times \vec{F}) = 0$ (Thm 15.12)

④ $\nabla \cdot (\vec{F} + \vec{G}) = \nabla \cdot \vec{F} + \nabla \cdot \vec{G}$

$\nabla \cdot (c\vec{F}) = c(\nabla \cdot \vec{F})$ c is a constant

$\nabla \cdot (u\vec{F}) = \nabla u \cdot \vec{F} + u(\nabla \cdot \vec{F})$, u is a scalar function

⑤ $\nabla \times (\vec{F} + \vec{G}) = \nabla \times \vec{F} + \nabla \times \vec{G}$

$\nabla \times (c\vec{F}) = c(\nabla \times \vec{F})$ c is a constant

⑥ Conditions: each component of \vec{F} has continuous second partial derivative on open connected region R . Then \vec{F} is conservative $\Leftrightarrow \nabla \times \vec{F} = \vec{0}$ (Thm 15.3 Page 1113)

Topic 2. Green's theorem

Conditions: R is a connected and simply connected 2D region, and its boundary is curve C which is simple closed and piecewise-smooth. Let $\vec{F} = \langle f, g \rangle$.

Circulation Form Thm 15.7. $\oint_C \vec{F} \cdot d\vec{r} = \oint_C f dx + g dy = \iint_R 2D \operatorname{curl} \vec{F} dA = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$

Flux Form Thm 15.8. $\oint_C \vec{F} \cdot \vec{n} ds = \oint_C f dy - g dx = \iint_R 2D \operatorname{div} \vec{F} dA = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA$

where C is counterclockwise oriented.

Applications

① If the line integral is difficult to evaluate ($\oint_C \vec{F} \cdot d\vec{r}$), then we could use green's thm

to calculate corresponding double integral.

② To calculate the area of an irregular region $\iint_R 1 dA$, we could calculate the line integral on the boundary using one of the three formulas.

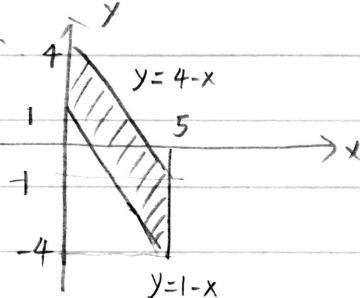
$$\iint_R 1 dA = \oint_C x dy = -\oint_C y dx = \frac{1}{2} \oint_C x dy - y dx$$

Examples:

HW II #36 $\vec{F} = \langle x-y, -x+3y \rangle$ R: parallelogram $\{(x,y) \mid -x \leq y \leq 4-x, 0 \leq x \leq y\}$

① $\oint_C \vec{F} \cdot d\vec{r}$ C is the boundary counterclockwise

Answer



Method I: Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ directly.

You have to parametrize form of boundary.

4 line segments!

Method II: Green's thm

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \iint_R 2 \operatorname{curl} \vec{F} dA = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA = \iint_R (1+1) dA = 0 \\ &= \iint_R 2 dA \quad \cancel{\iint_R \int_{1-x}^{4-x} 2 dy dx} \end{aligned}$$

② $\oint_C \vec{F} \cdot \vec{n} ds$

Answer: Use green's thm!

$\rightarrow = 4 \cdot \underline{\text{Area of } R}$

$$\oint_C \vec{F} \cdot \vec{n} ds = \iint_R \nabla \cdot \vec{F} dA = \iint_R \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} dA = \iint_R 4 dA = \int_0^5 \int_{1-x}^{4-x} 4 dy dx$$

More practice: #35. #37 #34

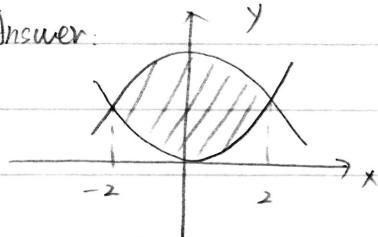
#35 $\vec{F} = \langle x, y \rangle$ R: half annulus $5 \leq r \leq 7$

$$\oint_C \vec{F} \cdot \vec{n} ds = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA = \iint_R 2 dA = \int_0^{\pi} \int_5^7 2 r dr d\theta$$

HW II #31: Find the area bounded by $\vec{r}(t) = \langle t, 6t^2 \rangle$ and $\vec{r}(t) = \langle t, 28-t^2 \rangle$ $-2 \leq t \leq 2$

Answer:

using Line integral.



Answer: $\iint_R \mathbf{F} \cdot d\mathbf{A} = \oint_C \mathbf{F} \times d\mathbf{y} = \int_{C_1} \mathbf{F} \times dy + \int_{C_2} \mathbf{F} \times dy$

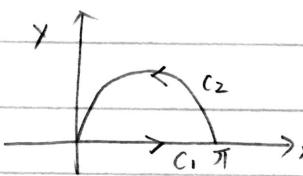
C₁: $\vec{r}(t) = \langle t, 6t^2 \rangle$ t is from -2 to 2

C₂: $\vec{r}(t) = \langle t, 28-t^2 \rangle$ t is from 2 to -2 !

odd $\int_{C_1} \mathbf{F} \times dy = \int_{-2}^2 \langle 0, 12t \rangle dt = \int_{-2}^2 12t^2 dt$

$\int_{C_2} \mathbf{F} \times dy = \int_2^{-2} \langle 0, (-2t) \rangle dt = \int_2^{-2} -2t^2 dt = \int_2^{-2} -2(-t)^2 (-dt) = \int_{-2}^2 2t^2 dt$

Note: HW 11 # 29.



C₁: $\vec{r}(t) = \langle 0, t \rangle$ t is from 0 to π

C₂: $\vec{r}(t) = \langle t, \sin t \rangle$ t is from π to 0

$\vec{F} = \langle 4y, 2x \rangle$

$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{\pi}^0 \langle 4\sin t, 2t \rangle \cdot \langle 1, \cos t \rangle dt$

$= \int_{\pi}^0 4\sin t + 2t \cos t dt$

$= \int_0^{\pi} 4\sin(\pi-t) + 2(\pi-t) \cos(\pi-t) (-dt)$

$= \int_0^{\pi} 4\sin t + 2(\pi-t) \cos t dt$

↓ $\begin{matrix} \text{change of var} \\ t = \pi - t \end{matrix}$

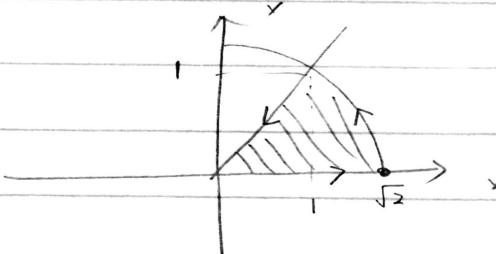
Practice: HW 11 # 30

Examples from Final Dec 2016

#6 $R = \{(x,y) | x \geq 0, y \geq 0, y \leq x, y \leq 2-x^2\}$ boundary C. counterclockwise

$\vec{F} = \langle x^2y + e^{\sin(x^2)}, \frac{x^3}{3} + x^2 + e^{\cos(y^2)} \rangle$

(a) Sketch R



(b) Compute curl of F

$\text{curl } \vec{F} = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = x^2 + 2x - x^2 = 2x$

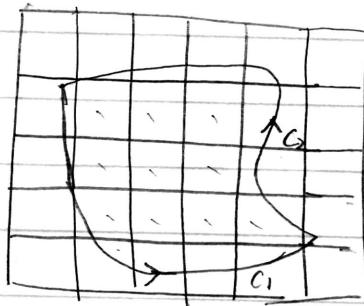
(c) Find circulation of \vec{F} around boundary of R

$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl } \vec{F} dA = \iint_R 2x dA = \int_0^1 \int_y^{\sqrt{2-y}} 2x dx dy$

$= \int_0^1 x^2 \Big|_y^{\sqrt{2-y}} dy = \int_0^1 2-y-y^2 dy = 2y - \frac{1}{2}y^2 - \frac{1}{3}y^3 \Big|_0^1$

$= 2 - \frac{1}{2} - \frac{1}{3} = \frac{7}{6}$

#7 (Similar to #8 Spring 2017 Final)



- (a) How many pieces should C be parameterized? 2
- (b) Approximately what is the area of R? 12 or 13
- (c) Suppose \vec{F} is a vector field with $\nabla \cdot \vec{F} = -12$. Approximate $\oint_C \vec{F} \cdot \vec{n} ds$ on R

Answer: $\oint_C \vec{F} \cdot \vec{n} ds = \iint_R \nabla \cdot \vec{F} dA = \iint_R -12 dA = -12 \cdot \text{Area of } R$ #

- (d) Suppose $\vec{F} = \langle e^x, 2x \rangle$ and $\oint_{C_1} \vec{F} \cdot d\vec{r} = 33$. Find $\oint_{C_2} \vec{F} \cdot d\vec{r}$?

Answer: $\oint_{C_1} \vec{F} \cdot d\vec{r} = \iint_R \text{curl } \vec{F} dA = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA = \iint_R 2 dA$

$$\oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r}$$

$$\Rightarrow \oint_{C_2} \vec{F} \cdot d\vec{r} = \iint_R \text{curl } \vec{F} dA - \oint_{C_1} \vec{F} \cdot d\vec{r}$$

$$= 2 \cdot \text{Area of } R + 33 \quad \#$$

Remark:

- (1) Given $\vec{F} = \langle f, g, h \rangle$, find φ such that $\vec{F} = \nabla \varphi$

- (2) Change of var.

$$\int_0^{2\pi} 2 \cos t \sin t \sqrt{1 + \sin^2 t} dt$$

$$u = \underline{1 + \sin^2 t} \quad du = 2 \sin t \cos t dt$$

$$\int \frac{\sqrt{2}}{3} \sqrt{u} du = \frac{\sqrt{2}}{3} u^{\frac{3}{2}} \rightarrow \text{range for } u: [1, 2]$$

But cannot plug in [1, 2]

