

§15.2 - 15.3

Topic 1: Line integral for vector field.

Note: $\int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F}(t) \cdot \vec{r}'(t) dt$ where $\vec{r}(t)$ is the parametric form of C

$$\text{Assume } \vec{F} = \langle f(t), g(t), h(t) \rangle \quad \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\begin{aligned} \text{Then } \int_a^b \vec{F} \cdot \vec{r}' dt &= \int_a^b f(t)x'(t) + g(t)y'(t) + h(t)z'(t) dt \\ &= \int_a^b f(t)x'(t) dt + \int_a^b g(t)y'(t) dt + \int_a^b h(t)z'(t) dt \end{aligned}$$

$$\boxed{\text{Alternative form}} \quad = \int_a^b f(t) dx(t) + \int_a^b g(t) dy(t) + \int_a^b h(t) dz(t)$$

Def: Circulation: $\int_C \vec{F} \cdot \vec{T} ds$ if C is a closed smooth oriented curve

Def: Flux: $\vec{F} = \langle f, g \rangle \quad \vec{r} = \langle x(t), y(t) \rangle \quad a \leq t \leq b$

$\int_C \vec{F} \cdot \vec{n} ds = \int_a^b f(t)y'(t) - g(t)x'(t) dt$, where C is a smooth oriented curve that does not intersect itself.

$$\left. \begin{aligned} &\text{Derivation: } \vec{n} = \vec{r} \times \vec{k} \quad \text{where } \vec{r} = \langle T_x, T_y, 0 \rangle = \langle \frac{x'}{|\vec{r}'|}, \frac{y'}{|\vec{r}'|}, 0 \rangle \\ &\Rightarrow \vec{n} = \frac{\langle y'(t), -x'(t) \rangle}{|\vec{r}'(t)|} \quad (1) \\ &ds = |\vec{r}'(t)| dt \quad (2) \\ &(1)(2) \Rightarrow \int_C \vec{F} \cdot \vec{n} ds = \int_a^b (f y' - g x') dt \end{aligned} \right)$$

Examples: HW 11 #18

$$\vec{F} = \langle by, cx \rangle, C: x^2 + y^2 = 1 \text{ counterclockwise}$$

(1) Find a, b such that $\int_C \vec{F} \cdot \vec{T} ds = 0$

$$\text{Answer: } \vec{r}(t) = \langle \cos t, \sin t \rangle \quad 0 \leq t \leq 2\pi$$

$$\vec{F}(t) = \langle b \sin t, c \cos t \rangle$$

$$\vec{r}'(t) = \langle -\sin t, \cos t \rangle = b \left(\frac{1 - \cos 2t}{2} \right) + c \left(\frac{\cos 2t - 1}{2} \right)$$

$$\vec{F} \cdot \vec{r}' = \underbrace{-b \sin^2 t + c \cos^2 t}_{= C - (b+c) \sin^2 t} = -c \cos^2 t + c \sin^2 t - c \sin^2 t - b \sin^2 t$$

$$= C - (b+c) \sin^2 t = C - (b+c) \frac{1 - \cos(2t)}{2} = \frac{C-b}{2} + \frac{1}{2}(b+c) \cos(2t)$$

$$\int_0^{2\pi} \vec{F} \cdot \vec{r}' dt = \int_0^{2\pi} \frac{C-b}{2} + \frac{1}{2}(b+c) \cos(2t) dt$$

$$= \left(\frac{C-b}{2} \cdot (2\pi) + \frac{1}{4}(b+c) \sin(2t) \right) \Big|_0^{2\pi}$$

$$= C-b = 0$$

(2) Find a, b such that $\int_C \vec{F} \cdot \vec{n} ds = 0$

$$\text{Answer. } \int_C \vec{F} \cdot \vec{n} ds = \int_0^{2\pi} f \cdot y' - g \cdot x' dt$$

$$= \int_0^{2\pi} a \cos t + d \sin^2 t dt \quad a \cdot \frac{\cos t - 1}{2} + d \cdot \frac{1 - \cos 2t}{2}$$

$$= \int_0^{2\pi} a \cos^2 t + a \sin^2 t + (d-a) \sin^2 t dt$$

$$= \int_0^{2\pi} a + (d-a) \left(\frac{1 - \cos 2t}{2} \right) dt$$

$$= \int_0^{2\pi} \frac{d+a}{2} - \frac{d-a}{2} \cos 2t dt \quad \checkmark$$

$$= \frac{d+a}{2}(2\pi) - \frac{1}{2}(d-a) \sin 2t \Big|_0^{2\pi}$$

$$= d+a = 0$$

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Topic 2 Conservative vector field

(1) The followings are equivalent

① \vec{F} is conservative, i.e. $\vec{F} = \nabla \varphi$ on region R

② Path independence (Thm 15.4/15.5), i.e. $\int_C \vec{F} \cdot d\vec{r} = \varphi(B) - \varphi(A)$ for all points A and B and

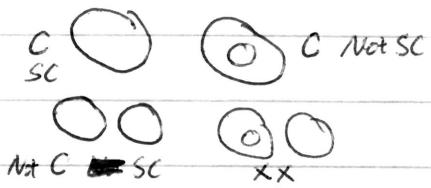
all piecewise smooth oriented curves C from A to B

③ $\oint_C \vec{F} \cdot d\vec{r} = 0$ for all simple piecewise-smooth closed oriented curves C in R .

④ Let $F = \langle f, g, h \rangle$, $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, $\frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}$, $\frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$ on a connected and simply connected region D .

Def: connected, closed, simple curves

connected, simply connected region



(2) Given a conservative vector field $\vec{F} = \langle f, g, h \rangle$. Find φ such that $\vec{F} = \nabla \varphi$.

$$\text{Step 1: } \varphi = \int \varphi_x dx + C(y, z) = \int f dx + C(y, z)$$

\triangle Simply connected

any curve closed curve
can be deformed and
connected to a point
in R .

$$\text{Step 2: Compute } \varphi_y \text{ and equate it to } g \text{ to obtain } C_y(y, z)$$

$$\text{Step 3: } C_y(y, z) = \int C_y(y, z) dy + d(z)$$

$$\text{Update } \varphi = \int f dx + \int C_y dy + d(z)$$

\triangle Connected

connect any two
point in R

$$\text{Step 4: Compute } \varphi_z \text{ and equate it to } h \text{ to obtain } d'(z)$$

$$\text{Step 5: } d(z) = \int d'(z) dz + C$$

$$\text{update } \varphi = \int f dx + \int C_y dy + \int d'(z) dz + C$$

Example: Practice HWII #20.21

$$\vec{F} = \langle z, 1, x \rangle \text{ on } \mathbb{R}^3$$

Is \vec{F} conservative? If it is, find φ such that $\vec{F} = \nabla \varphi$

Answer. $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = 0 \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x} = 1 \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y} = 0 \quad \text{Yes!}$

Step 1. $\varphi = \int g_x dx + C(y, z) = \int f dx + C(y, z) = \int z dx + C(y, z) = xz + C(y, z)$

Step 2. $\varphi_y = \frac{\partial}{\partial y} [xz + C(y, z)] = \frac{\partial}{\partial y} C(y, z) = g = 1$

Step 3. $C(y, z) = \int 1 dy + d(z) = y + d(z)$

$$\varphi = xz + y + d(z)$$

Step 4. $\varphi_z = \frac{\partial}{\partial z} [xz + y + d(z)] = x + d'(z) = h = x \Rightarrow d'(z) = 0$

Step 5. $d(z) = \int 0 dz + C = C$

$$\varphi = xz + y + C$$

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Example: $\vec{F} = \langle x^3 + xy, \frac{x^2}{2} + y \rangle \text{ on } \mathbb{R}^3$

Answer. $\frac{\partial f}{\partial y} = x = \frac{\partial g}{\partial x} \quad \text{Yes!}$

Step 1. $\varphi = \int g_x dx + C(y) = \int x^3 + xy dx + C(y) = \frac{1}{4}x^4 + \frac{1}{2}x^2y + C(y)$

Step 2. $\varphi_y = \frac{1}{2}x^2 + C'(y) = g = \frac{1}{2}x^2 + y \Rightarrow C'(y) = y$

Step 3. $C(y) = \int y dy + C = \frac{1}{2}y^2 + C$

$$\varphi = \frac{1}{4}x^4 + \frac{1}{2}x^2y + \frac{1}{2}y^2 + C$$

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Example: Use two different method to evaluate $\int_C \nabla \varphi \cdot d\vec{r}$

Practice
HWII #22

$$\varphi = 3x + 4y \quad \vec{r} = \langle 3-t, t \rangle \quad 0 \leq t \leq 3$$

Answer: Method 1:

$$\nabla \varphi = \langle 3, 4 \rangle \quad \vec{r}'(t) = \langle -1, 1 \rangle \quad \nabla \varphi \cdot \vec{r}' = -3 + 4 = 1$$

$$\int_C \nabla \varphi \cdot d\vec{r} = \int_0^3 \nabla \varphi \cdot \vec{r}' dt = \int_0^3 1 dt = 3$$

Method 2:

$F = \nabla \varphi$ is conservative.

$$\int_C \nabla \varphi \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} = \varphi(B) - \varphi(A) = 3 \times 0 + 4 \times 3 - 3 \times 3 - 4 \times 0 = 3$$

Note: $t=3 \Rightarrow B(0, 3)$

$t=0 \Rightarrow A(3, 0)$

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Problems from Finals

1. $\vec{F} = \langle zxy, x^2z, y+1 \rangle$ on \mathbb{R}^3

② Is \vec{F} conservative? If it is, find φ

Answer: $\frac{\partial f}{\partial y} = 2x = \frac{\partial g}{\partial y}$

$$\frac{\partial f}{\partial z} = 0 = \frac{\partial h}{\partial x} \quad \checkmark$$

$$\frac{\partial g}{\partial z} = 1 = \frac{\partial h}{\partial y}$$

Step 1. $\varphi = \int \varphi_x dx + C(y, z) = \int zxy dx + C(y, z) = x^2y + C(y, z)$

Step 2: $\varphi_y = \frac{\partial}{\partial y}(x^2y + C(y, z)) = x^2 + \frac{\partial}{\partial y}C(y, z) = g = x^2 + z$

$$\frac{\partial}{\partial y}(C(y, z)) = z$$

Step 3: $C(y, z) = \int z dy + d(z) = yz + d(z)$

$$\varphi = x^2y + yz + d(z)$$

Step 4: $\varphi_z = y + d'(z) = h = y + 1 \Rightarrow$

$$d'(z) = 1$$

Step 5 $d(z) = \int 1 dz + C = z + C$

$$\varphi = x^2y + yz + z + C$$

⑥ Evaluate $\int_C \vec{F} \cdot d\vec{r}$ along line segment $P(0, 0, 1)$ to $Q(1, 1, 1)$

Answer since $\vec{F} = \nabla \varphi$, $\int_C \vec{F} \cdot d\vec{r} = \varphi(B) - \varphi(A)$

$$= 3 + C - (1 + C)$$

$$= 2$$

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2 $\vec{F} = \langle -ysm(xy), -xsm(xy) + 6y \rangle$ on \mathbb{R}^2

② Is \vec{F} conservative? If it is, find φ .

Answer: $\frac{\partial f}{\partial y} = -sm(xy) - xycos(xy) \Rightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \quad \checkmark$

$$\frac{\partial f}{\partial x} = -sm(xy) - xycos(xy)$$

Step 1: $\varphi = \int \varphi_x dx + C(y) = \int -ysm(xy) dx + C(y) = -cos(xy) + C(y)$

Step 2: $\varphi_y = \frac{\partial}{\partial y}(-cos(xy) + C(y)) = -xsm(xy) + C'(y) = -xsm(xy) + 6y$

$$\Rightarrow C'(y) = 6y$$

Step 3 $C(y) = \int 6y dy + C = 3y^2 + C$

$$\varphi = -cos(xy) + 3y^2 + C$$

⑥ C is the ellipse $x^2 + 3y^2 = 1$ with counterclockwise. What is the value $\oint_C \vec{F} \cdot d\vec{r}$. Justify your answer.

Answer. $\oint_C \vec{F} \cdot d\vec{r} = 0$, since \vec{F} is conservative and C is simple closed.