

§ 15.1 - 15.2

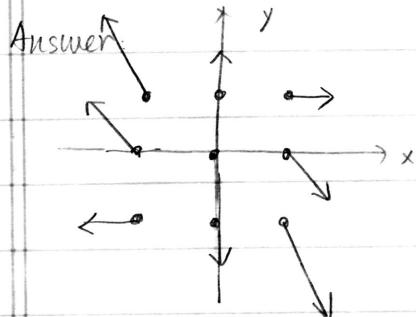
Topic 2: Vector field.

Def. A vector field $\vec{F} = \langle f(x,y,z), g(x,y,z), h(x,y,z) \rangle$ assigns each point (x,y,z) a vector $\langle f(x,y,z), g(x,y,z), h(x,y,z) \rangle$.

Def. Let φ be a given function, $F = \nabla\varphi$ is called gradient field and φ is called potential function of F .

Examples:

- (1) Sketch the vector field $\vec{F} = \langle x, y-x \rangle$



Points	Vectors	Points	Vectors
$(0,0)$	$\langle 0,0 \rangle$	$(1,1)$	$\langle 1,0 \rangle$
$(1,0)$	$\langle 1,-1 \rangle$	$(1,-1)$	$\langle 1,2 \rangle$
$(0,1)$	$\langle 0,1 \rangle$	$(-1,1)$	$\langle -1,2 \rangle$
$(-1,0)$	$\langle -1,1 \rangle$	$(-1,-1)$	$\langle -1,0 \rangle$
$(0,-1)$	$\langle 0,-1 \rangle$		

- (2) Let $\varphi = y - x^2$, $0 \leq x \leq 1$

① Find gradient field $\vec{F} = \nabla\varphi = \langle -2x, 1 \rangle$

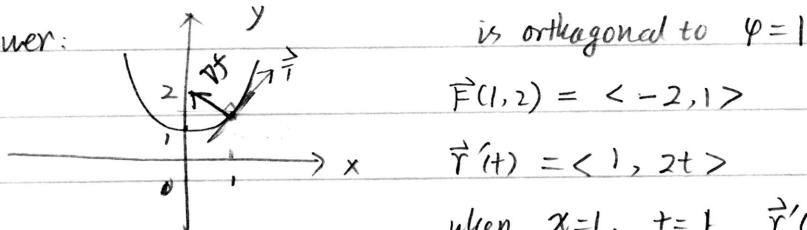
② Find a parametric form of $\varphi = 1$ in the form of $\vec{r}(t) = \langle x(t), y(t) \rangle$

$$\varphi = 1 \Leftrightarrow y - x^2 = 1 \Leftrightarrow y = 1 + x^2$$

$$\vec{r}(t) = \langle t, 1+t^2 \rangle \quad 0 \leq t \leq 1$$

- ③ Sketch the gradient vector at $(1,2)$ and verify that the gradient vector

Answer:



It's easy to see that $\vec{F}(1,2) \cdot \vec{r}'(1) = 0$. So \vec{F} is orthogonal to $\varphi = 1$ at $(1,2)$.

Topic 2: Line integrals.

Type I: Scalar line integral.

Goal: Evaluate $\int_C f ds$

Application
 ① Arc length $L = \int_C ds$
 ② Average value $\bar{f} = \frac{\int_C f ds}{L}$

Step 1: Find the parametric form of C : $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ $a \leq t \leq b$

Step 2: Calculate $|\vec{r}'(t)|$

Step 3: $\int_C f ds = \int_a^b f(x(t), y(t), z(t)) \cdot |\vec{r}'(t)| dt$

| Example:

$\int_C (2x - 3y) ds$: C : line segment from $(1, 0)$ to $(0, 1)$

Answer: Step 1: $\vec{r}(t) = (1, 0) + t \cdot \langle 1, 1 \rangle = \langle 1+t, t \rangle$ $0 \leq t \leq 1$

$$\text{Step 2: } |\vec{r}'(t)| = \sqrt{1+1} = \sqrt{2}$$

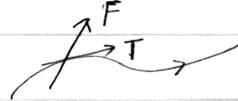
$$\begin{aligned} \text{Step 3: } \int_C f ds &= \int_0^1 [2(1+t) - 3t] \sqrt{2} dt \\ &= \int_0^1 (-2-t) \sqrt{2} dt \end{aligned}$$

$$= -2\sqrt{2}t - \frac{\sqrt{2}}{2}t^2 \Big|_0^1$$

$$= -2\sqrt{2} - \frac{\sqrt{2}}{2} = -\frac{5\sqrt{2}}{2} \quad \#$$

Type II: Line integral of vector field.

Goal: Evaluate $\int_C \vec{F} \cdot \vec{T} ds$



Derivative formula:

$$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \quad ds = |\vec{r}'(t)| dt$$

$$\int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F} \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|} dt = \int_a^b \vec{F} \cdot \vec{r}'(t) dt$$

Alternative formula:

Let $\vec{F}(t) = \langle f(t), g(t) \rangle$ and $\vec{r}(t) = \langle x(t), y(t) \rangle$

$$\vec{F} \cdot \vec{r}'(t) = \langle f(t), g(t) \rangle \cdot \langle x'(t), y'(t) \rangle = f(t)x'(t) + g(t)y'(t)$$

$$\int_a^b \vec{F} \cdot \vec{r}'(t) dt = \int_a^b f(t)x'(t) dt + \int_a^b g(t)y'(t) dt = \int_C f dx + g dy$$

Example: $\vec{F} = \langle y, x \rangle$ $C: x^2 + y^2 = 4$ counterclockwise

Find $\int_C \vec{F} \cdot \vec{T} ds$

Answer: $\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$ $0 \leq t \leq 2\pi$

$$\int_C \vec{F} \cdot \vec{T} ds = \int_0^{2\pi} \vec{F}(t) \cdot \vec{r}'(t) dt$$

$$\vec{F}(t) = \langle y(t), x(t) \rangle = \langle 2\sin t, 2\cos t \rangle$$

$$\vec{r}'(t) = \langle -2\sin t, 2\cos t \rangle$$

$$\vec{F}(t) \cdot \vec{r}'(t) = -4\sin^2 t + 4\cos^2 t = 8\cos 2t$$

$$\int_0^{2\pi} \vec{F} \cdot \vec{r}' dt = \int_0^{2\pi} 8\cos 2t dt = 4\sin 2t \Big|_0^{2\pi} = 0$$

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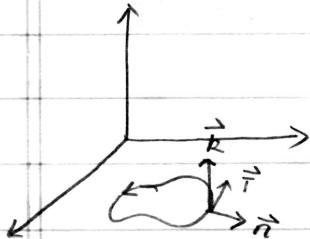
Topic 3: Circulation and Flux of a vector field.

Def. If C is a closed smooth oriented curve. $\oint_C \vec{F} \cdot \vec{T} ds$ is the circulation of \vec{F} on C

Def: If C is a smooth oriented curve that does not intersect itself.

$\oint_C \vec{F} \cdot \vec{n} ds$ is the flux of vector field of \vec{F} on C .

Calculation of flux:



$$\vec{n} = \vec{T} \times \vec{k} \quad \text{and} \quad \vec{k} = \langle 0, 0, 1 \rangle$$

$$\vec{T} = \langle T_x, T_y, 0 \rangle \text{ where } T_x = \frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}$$

$$T_y = \frac{y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}$$

$$\vec{n} = \frac{\langle x'(t), y'(t) \rangle}{\sqrt{x'(t)^2 + y'(t)^2}}$$

$$\text{Hence } \oint_C \vec{F} \cdot \vec{n} ds = \int_a^b [\vec{F}(t) \cdot \vec{y}'(t) - g(t)x'(t)] dt$$

Examples:

$$\vec{F} = \langle y, x \rangle \quad C: x^2 + \frac{y^2}{4} = 1 \text{ counterclockwise}$$

Find circulation $\oint_C \vec{F} \cdot \vec{T} ds$ and flux $\oint_C \vec{F} \cdot \vec{n} ds$

Answer: Parametric form of C : $\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle \quad 0 \leq t \leq 2\pi$

$$\vec{F} \cdot \vec{r}'(t) = \langle 2\sin t, 2\cos t \rangle \cdot \langle -\sin t, 2\cos t \rangle = 2\cos^2 t - 2\sin^2 t = 4\cos 2t$$

$$\oint_C \vec{F} \cdot \vec{T} ds = \int_0^{2\pi} \vec{F} \cdot \vec{r}'(t) dt = \int_0^{2\pi} 4\cos 2t dt = 0$$

$$\begin{aligned} \oint_C \vec{F} \cdot \vec{n} ds &= \int_0^{2\pi} [2\sin t \cdot 2\cos t - 2\cos t \cdot (-\sin t)] dt = \int_0^{2\pi} 5\cos t \sin t dt \\ &= \frac{5}{4} \sin 2t \Big|_0^{2\pi} = 0 \end{aligned}$$

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Example: $\oint_C (y - z) ds \quad C: \vec{r}(t) = \langle 3\cos t, 3\sin t, t \rangle$

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$$\text{Answer: } \vec{r}(t) = \langle -3\sin t, 3\cos t, t \rangle \quad |\vec{r}(t)| = \sqrt{9 + t^2} = \sqrt{10}$$

$$\int_0^{2\pi} (3\sin t - t) \sqrt{10} dt = [3\cos t - \frac{1}{2}t^2] \Big|_0^{2\pi} = -2\pi^2 \sqrt{10} \#$$