

§ 14.5.

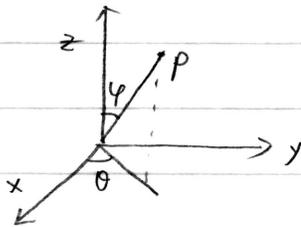
Goal: $\iiint_D f \, dv$

Cartesian: $\iiint_D f \, dv = \iint_{R_{xy}} \int_{g_1(x,y)}^{h(x,y)} f(x,y,z) \, dz \, dA$ and the other two forms

Cylindrical: $\iiint_D f \, dv = \iint_{R_{xy}} \int_{g_1(r \cos \theta, r \sin \theta)}^{h(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) \, dz \cdot r \, dr \, d\theta$

$$= \int_a^b \int_{r_1(\theta)}^{r_2(\theta)} \int_{g_1(r, \theta)}^{h(r, \theta)} f(r \cos \theta, r \sin \theta, z) \, dz \cdot r \, dr \, d\theta$$

Spherical: $\iiint_D f \, dv = \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \int_{g_1(\theta, \varphi)}^{h(\theta, \varphi)} f(\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi) \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta$



$$x = \rho \sin \varphi \cos \theta$$

$$0 \leq \theta \leq 2\pi$$

$$y = \rho \sin \varphi \sin \theta$$

$$0 \leq \varphi \leq \pi$$

→ HW10 #26

$$z = \rho \cos \varphi$$

Equations of some common figures in spherical coordinates

- (a) Sphere with center $(0, 0, 0)$ and radius a : $\rho = a$
- (b) Sphere with center $(0, 0, a)$ and radius a : $\rho = 2a \cos \varphi$ $0 \leq \varphi \leq \frac{\pi}{2}$
- (c) Horizontal plane $z = a$: $\rho = a \sec \varphi$ $0 \leq \varphi < \frac{\pi}{2}$ or $\frac{\pi}{2} < \varphi \leq \pi$.
- (d) Cone: $\varphi = \varphi_0$
- (e) Cylinder $x^2 + y^2 = a^2$: $\rho = a \csc \varphi$.

Applications of integration: (§ 14.6)

(1) Volume $V = \iiint_D 1 \, dv$

(2) Mass $M = \iiint_D \rho \, dv$ ρ density: density function

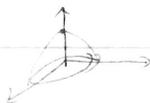
(3) Moment $M_{yz} = \iiint_D x \rho \, dv$

(4) Center $\bar{x} = \frac{M_{yz}}{M}$

Examples:

HW10 #17 Find the mass of the solid paraboloid $D = \{(r, \theta, z) : 0 \leq z \leq 25 - r^2, 0 \leq r \leq 5\}$ with density $\rho(r, \theta, z) = 1 + \frac{z}{25}$.

Answer: $M = \iiint_D \rho \, dv = \iint_{R_{xy}} \int_{g_1(r, \theta)}^{h(r, \theta)} \rho(r, \theta, z) \, dz \, r \, dr \, d\theta$



$$= \int_0^{2\pi} \int_0^5 \int_0^{25-r^2} \left(1 + \frac{z}{25}\right) \, dz \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^5 \left. z + \frac{z^2}{50} \right|_0^{25-r^2} r dr d\theta$$

$$= \int_0^{2\pi} \int_0^5 \left[(25-r^2) + \frac{(25-r^2)^2}{50} \right] r dr d\theta$$

$$= \dots$$

✓ #16

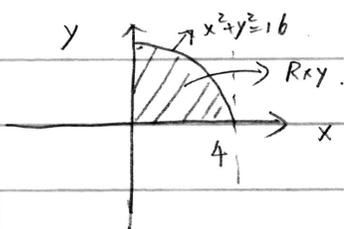
$$\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{x^2+y^2}} (x^2+y^2)^{-\frac{1}{2}} dz dy dx \rightarrow dz r dr d\theta$$

\downarrow R_{xy} \downarrow rewrite in terms of r, θ \downarrow $r dr d\theta$
 \downarrow Polar

Practice

Review

Prob1 #43



$$= \int_0^{\frac{\pi}{2}} \int_0^4 \int_0^r (r^2)^{-\frac{1}{2}} dz \cdot r dr d\theta$$

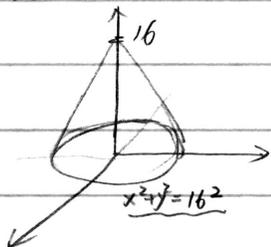
$$= \int_0^{\frac{\pi}{2}} \int_0^4 \int_0^r 1 dz dr d\theta$$

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✓ Final exam. Mass of solid cone bdd by $z = 16 - \sqrt{x^2 + y^2}$ and $z = 0$ given density $P = z$

Practice: Answer:

HW10 #17 #33
 #18 #34
 #19
 #27



Review Prob2

#55

$$M = \iiint_D P dv$$

$$= \iint_{R_{xy}} \int_0^{16-r} z dz dA$$

$$= \int_0^{2\pi} \int_0^{16} \int_0^{16-r} z dz r dr d\theta$$

$$= \int_0^{2\pi} \int_0^{16} \frac{1}{2} (16-r)^2 \cdot r dr d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{8} (16-r)^4 - \frac{8}{3} (16-r)^3 \right]_0^{16} d\theta$$

Note: $\int \frac{1}{2} (16-r)^2 r dr$

$$u = 16-r \quad du = -dr$$

$$= \int \frac{1}{2} u^2 (16-u) (-du)$$

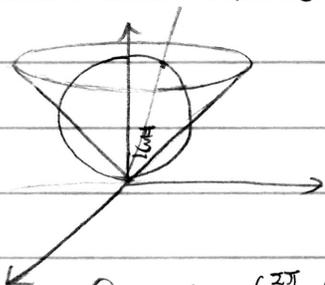
$$= \int \frac{1}{2} u^2 (u-16) du$$

$$= \frac{1}{8} u^4 - \frac{8}{3} u^3$$

$$= \frac{1}{8} (16-r)^4 - \frac{8}{3} (16-r)^3$$

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✓ HW10 #25. Cone $\varphi = \frac{\pi}{3}$ sphere $P = 4 \cos \varphi$



- (a) Volume inside the cone and ~~outside~~ ^{inside} the sphere
- (b) Volume outside cone and inside the sphere

Practice:

HW10 #24
 #29

Review Prob2

#49

$$(a) V = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^{4 \cos \varphi} P^2 \sin \varphi d\rho d\varphi d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \frac{1}{3} P^3 \sin \varphi \Big|_0^{4 \cos \varphi} d\varphi d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \frac{64}{3} (\cos \varphi)^3 \sin \varphi d\varphi d\theta \quad u = \cos \varphi \quad du = -\sin \varphi d\varphi$$

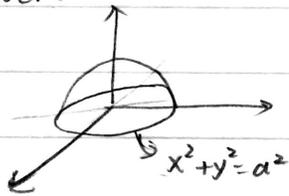
$$= \int_0^{2\pi} \cdot \frac{64}{3} \cdot \left(-\frac{1}{4}\right) (\cos \varphi)^4 \Big|_0^{\frac{\pi}{3}} d\theta$$

$$\begin{aligned}
 &= \int_0^{2\pi} \frac{16}{3} (\cos^4 0 - \cos^4 \frac{\pi}{3}) d\theta \\
 &= \int_0^{2\pi} \frac{16}{3} (1 - \frac{1}{16}) d\theta \\
 &= 10\pi
 \end{aligned}$$

$$\textcircled{b} \quad V = \int_0^{2\pi} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^{4\cos\varphi} \rho^2 \sin\varphi d\rho d\varphi d\theta$$

✓ Example: Center of hemisphere $x^2 + y^2 + z^2 = a^2$ with constant density 1.

Answer:



The figure is symmetric about xz -plane and yz -plane

The density is constant. $\Rightarrow \bar{x} = \bar{y} = 0$

Note: if density is not constant, it may not be true

$$\bar{z} = \frac{M_{xy}}{M}$$

$$M = \iiint_D 1 \, dv = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^a 1 \cdot \rho^2 \sin\varphi d\rho d\varphi d\theta = \frac{2\pi}{3} a^3$$

$$\begin{aligned}
 M_{xy} &= \iiint_D \bar{z} \, dv = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^a \rho \cos\varphi \cdot \rho^2 \sin\varphi d\rho d\varphi d\theta \\
 &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{1}{4} \rho^4 \sin\varphi \cos\varphi \Big|_0^a d\varphi d\theta \\
 &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{1}{8} a^4 \sin 2\varphi d\varphi d\theta \\
 &= \int_0^{2\pi} -\frac{1}{16} a^4 \cos 2\varphi \Big|_0^{\frac{\pi}{2}} d\theta \\
 &= \frac{a^4}{8} 2\pi
 \end{aligned}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\frac{\pi}{4} \frac{3}{2\pi} \cdot a^4}{\frac{2\pi}{3} a^3} = \frac{1}{8} a \quad \#$$

Compare with HW10 #36. (Cylindrical coordinates)