Math 2162.02
Autumn 2017
Exam 2
10/11/2017
Time Limit: 55 Minutes

Name: ______________________
Name.##: ______________________
Lecturer: Jenny Sheldon
TA: Yanli Wang
Recitation Time: ______________

This exam contains 7 pages (including this cover page) and 6 questions. The total number of points is 100. Calculators are not permitted, and the exam is closed book and closed notes.

Do not open the exam until instructed to do so. Show all your work; solutions without appropriate supporting details may not be given credit. You need not simplify purely numerical expressions for your final answer unless otherwise instructed. Do not cheat.

Question	Points	Score
1	16	
2	17	
3	18	
4	18	
5	16	
6	15	
Total:	100	

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1. (16 points) (a) (8 points) Determine the value of $\lim_{(x,y)\to(3,3)} \frac{x^2 - y^2}{x - y}$.

Answer:

$$\lim_{(x,y)\to(3,3)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(3,3)} \frac{\cancel{(x-y)}(x+y)}{\cancel{x-y}} = \lim_{(x,y)\to(3,3)} x+y = 6$$

(b) (8 points) Show that $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2}$ does not exist.

Answer:

For all linear paths $y = mx$

$$\lim_{x\to 0} \frac{x^3 \cdot mx}{x^6 + (mx)^2} = \lim_{x\to 0} \frac{mx^4}{x^2(x^4+m^2)} = \lim_{x\to 0} \frac{mx^2}{x^4+m^2} = 0$$

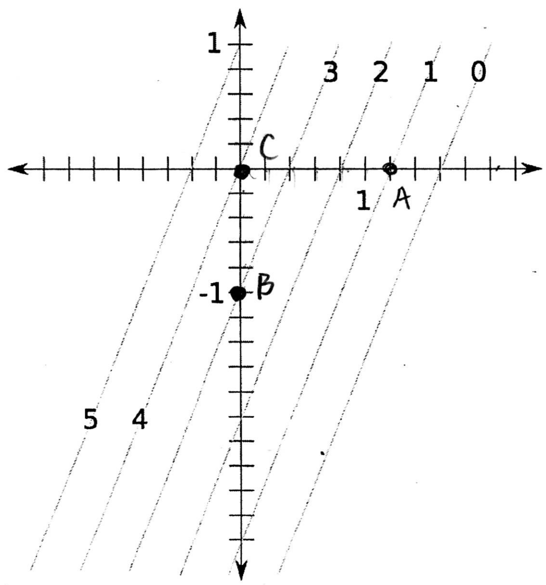
For cubic paths $y = mx^3$

$$\lim_{x\to 0} \frac{x^3 \cdot mx^3}{x^6 + (mx^3)^2} = \lim_{x\to 0} \frac{mx^6}{x^6 + m^2x^6} = \lim_{x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$

By two path test, the limit does not exist.

Note: You can either choose two different cubic paths or choose one linear path and one cubic path. Two linear paths won't work.

2. (17 points) Some level curves of a plane P are given below. Is the plane Q given by $-20x + 8y - 8z = 1$ parallel to P, perpendicular to P, or neither?



(Hint: do you know any points on the plane P?)

Answer: We need to find the normal vector of plane P.

From the graph, we can find three points on plane P

① $z=1$ level curve passes $x=1.2$ $y=0$ $\Rightarrow$ $(1.2, 0, 1) = A$

② $z=3$ level curve passes $x=0$ $y=-1$ $\Rightarrow$ $(0, -1, 3) = B$

③ $z=4$ level curve passes $x=0$, $y=0$ $\Rightarrow$ $(0, 0, 4) = C$

$\vec{V}_1 = \overrightarrow{AB} = \langle -1.2, -1, 2 \rangle$

$\vec{V}_2 = \overrightarrow{AC} = \langle -1.2, 0, 3 \rangle$

$$\vec{n}_1 = \vec{V}_1 \times \vec{V}_2 = \begin{vmatrix} i & j & k \\ -1.2 & -1 & 2 \\ -1.2 & 0 & 3 \end{vmatrix} = -3\vec{i} + 1.2\vec{j} - 1.2\vec{k}$$

$$= \langle -3, 1.2, -1.2 \rangle$$

The normal vector of plane Q is $\vec{n}_2 = \langle -20, 8, -8 \rangle$

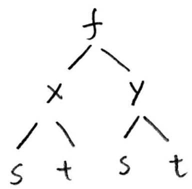
Obviously, $\vec{n}_2 = \frac{3}{20}\vec{n}_3$

So the two planes are parallel to each other.

3. (18 points) Let $f(x, y)$ be a differentiable function. Let $g(s, t) = f(2s + t, s^2 + t^2)$.

(a) (6 points) Write an expression for g_s in terms of f_x, f_y, s, and t.

Answer: $g(s,t) = f(x,y)$ where $x = 2s+t$, $y = s^2+t^2$



$$g_s = f_x x_s + f_y \cdot y_s$$
$$= f_x \cdot 2 + f_y \cdot 2s$$
$$= f_x(2s+t, s^2+t^2) \cdot 2 + f_y(2s+t, s^2+t^2) \cdot 2s$$

(b) (6 points) Write an expression for g_t in terms of f_x, f_y, s, and t.

Answer:

$$g_t = f_x x_t + f_y y_t$$
$$= f_x(2s+t, s^2+t^2) \cdot 1 + f_y(2s+t, s^2+t^2) \cdot 2t$$

(c) (6 points) Suppose you know that $f_x(6, 8) = 5$ and $f_y(6, 8) = 7$. Evaluate $g_s(2, 2)$ and $g_t(2, 2)$.

Answer:

Note: $t = 2$
$s = 2$

$$g_t(2,2) = f_x(6,8) \cdot 1 + f_y(6,8) \cdot 2 \cdot 2$$
$$= 5 \times 1 + 7 \times 4$$
$$= 5 + 28$$
$$= 33$$

4. (18 points) Consider the function (given in polar coordinates) $r = \cos(\theta)$ for $0 \le \theta \le \pi$.

(a) (4 points) Write a set of parametric equations $\vec{u}(t) = \langle f(t), g(t) \rangle$ for this curve.

Answer: $f(t) = r \cdot \cos\theta = \cos(\theta) \cdot \cos(\theta) = \cos^2(\theta)$

$g(t) = r \cdot \sin\theta = \cos(\theta) \cdot \sin(\theta)$

$\vec{u}(t) = \langle \cos^2(t), \cos(t)\sin(t) \rangle \quad 0 \le t \le \pi$

Other acceptable answers

① $r = \cos(\theta)$

$r^2 = r\cos(\theta)$

$x^2 + y^2 = x$

$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$

$f(t) = \frac{1}{2} + \frac{1}{2}\cos t$

$g(t) = \frac{1}{2}\sin t$

$\vec{u}(t) = \langle \frac{1}{2} + \frac{1}{2}\cos t, \frac{1}{2}\sin t \rangle$

$0 \le t \le 2\pi$

② From $x^2 + y^2 = x$

$y^2 = x - x^2$

$y = \pm\sqrt{x - x^2}$

$f(t) = t \quad g(t) = \pm\sqrt{t - t^2}$

$\vec{u}(t) = \langle t, \pm\sqrt{t - t^2} \rangle \quad 0 \le t \le 1$

(b) (8 points) Where, if anywhere, does this curve have vertical tangent lines?

Answer: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos^2 t - \sin^2 t}{-2\cos t \sin t}$

Vertical tangent line: $\begin{cases} \cos^2 t - \sin^2 t \neq 0 \\ -2\cos t \cdot \sin t = 0 \end{cases}$

$2\cos t \sin t = 0 \quad \sin(2t) = 0$

$2t = 0, \pi, 2\pi$

$t = 0, \frac{\pi}{2}, \pi$

Also, notice that when $t = 0, \frac{\pi}{2}, \pi$

$\cos^2 t - \sin^2 t \neq 0$

Conclusion: when $t = 0, \frac{\pi}{2}, \pi$, curve has vertical tangent line

(c) (6 points) Consider the curve $\vec{v}(t) = \langle f(t), g(t), t \rangle$. (Remember that f and g were defined in part (a). Where, if anywhere, does $\vec{v}(t)$ have vertical tangent lines?

Answer:

$\vec{v}'(t) = \langle f'(t), g'(t), 1 \rangle$

$= \langle -2\cos t \sin t, \cos^2 t - \sin^2 t, 1 \rangle$

$= \langle -\sin(2t), \cos(2t), 1 \rangle$

vertical tangent line for 3D curve $\Leftrightarrow$ $f'(t) = 0$ and $g'(t) = 0$

But $\begin{cases} \cos(2t) = 0 \\ \sin(2t) = 0 \end{cases}$ has no solution

Conclusion: the curve does not have vertical tangent line

5. (16 points) Suppose the velocity of a particle at time t is given by

$$\vec{v}(t) = \langle 3\sin t, 3\sqrt{5}\cos t, 6\sin t \rangle,$$

and its initial position is given by $\vec{r}(0) = \langle 0, 1, 2 \rangle$.

(a) (8 points) Find the length of the curve traced out by the particle from $t = 0$ to $t = \pi$.

Answer:

$$L = \int_0^{\pi} |\vec{r}\,'(t)|\,dt = \int_0^{\pi} |\vec{v}(t)|\,dt$$

$$= \int_0^{\pi} \sqrt{(3\sin t)^2 + (3\sqrt{5}\cos t)^2 + (6\sin t)^2}\,dt$$

$$= \int_0^{\pi} \sqrt{9\sin^2 t + 45\cos^2 t + 36\sin^2 t}\,dt$$

$$= \int_0^{\pi} \sqrt{45(\sin^2 t + \cos^2 t)}\,dt$$

$$= \int_0^{\pi} \sqrt{45}\,dt = \sqrt{45}\,\pi$$

(b) (8 points) Find the position function $\vec{r}(t)$.

Answer: $\vec{r}(t) = \int \vec{v}(t)\,dt + \vec{C}$

$$= \langle \int 3\sin t\,dt, \int 3\sqrt{5}\cos t\,dt, \int 6\sin t\,dt \rangle + \vec{C}$$

$$= \langle -3\cos t, 3\sqrt{5}\sin t, -6\cos t \rangle + \vec{C}$$

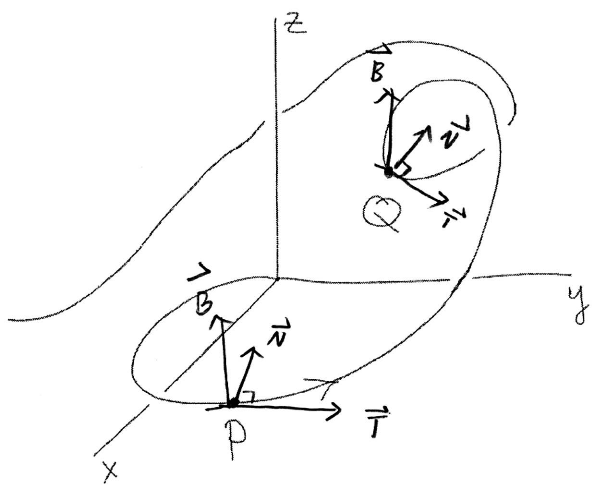
$$\vec{r}(0) = \langle -3, 0, -6 \rangle + \vec{C} = \langle 0, 1, 2 \rangle$$

$$\Rightarrow \vec{C} = \langle 0, 1, 2 \rangle - \langle -3, 0, -6 \rangle$$

$$= \langle 3, 1, 8 \rangle$$

$$\vec{r}(t) = \langle -3\cos t + 3, 3\sqrt{5}\sin t + 1, -6\cos t + 8 \rangle$$

6. (15 points) Consider the curve $\vec{r}(t)$ given below. Answer the following questions. No partial credit for the individual parts will be given on this page.



(a) (3 points) Sketch the vectors $\vec{T}$, $\vec{N}$, $\vec{B}$ at P.

(b) (3 points) Sketch the vectors $\vec{T}$, $\vec{N}$, $\vec{B}$ at Q.

Keys: $\vec{T}$ is tangent vector, pointing to positive orientation

$\vec{N} \perp \vec{T}$ and pointing to the inside of the curve

$\vec{B} = \vec{T} \times \vec{N}$. right hand rule

(c) (3 points) Is the curvature greater at P or at Q?

Q curvature measures the rate of change of tangent vector with respect to arc length parameter s, i.e. $|d\vec{T}/ds| = \kappa$.

(d) (3 points) Is the magnitude of the torsion greater at P or at Q?

Q torsion measures the rate that the curve twists out of the plane generated by $\vec{T}$ and $\vec{N}$.

(e) (3 points) What is $\vec{B} \times \vec{N}$?

$\vec{B} = \vec{T} \times \vec{N}$

$\Rightarrow \quad \vec{N} \times \vec{B} = \vec{T}$

$\Rightarrow \quad \vec{B} \times \vec{N} = -\vec{T}$