

Math 2162.02
Autumn 2017
Exam 1
9/13/2017
Time Limit: 55 Minutes

Name: _____
Name.##: _____
Lecturer: Jenny Sheldon
TA: Yanli Wang
Recitation Time: _____

This exam contains 7 pages (including this cover page) and 6 questions.
The total number of points is 100. Calculators are not permitted, and the exam is closed book and closed notes.

Do not open the exam until instructed to do so. Show all your work; solutions without appropriate supporting details may not be given credit. You need not simplify purely numerical expressions for your final answer unless otherwise instructed. Do not cheat.

Question	Points	Score
1	14	
2	16	
3	20	
4	20	
5	15	
6	15	
Total:	100	

Q2

1. (14 points) Find the interval of convergence for the power series $\sum_{k=1}^{\infty} \frac{(x+2)^k}{(k+1) \cdot 5^k}$.

Answer: Ratio Test.

$$\begin{aligned} \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| &= \lim_{k \rightarrow \infty} \left| \frac{(x+2)^{k+1}}{(k+2)5^{k+1}} \cdot \frac{(k+1) \cdot 5^k}{(x+2)^k} \right| \\ &= \lim_{k \rightarrow \infty} \left| \frac{(x+2)(k+1)}{(k+2) \cdot 5} \right| \\ &= \left| \frac{x+2}{5} \right| < 1 \end{aligned}$$

$$\begin{aligned} |x+2| &< 5 \\ -5 &< x+2 < 5 \\ -7 &< x < 3 \end{aligned}$$

- Check the endpoints

$$x = 3 \quad \sum_{k=1}^{\infty} \frac{5^k}{(k+1)5^k} = \sum_{k=1}^{\infty} \frac{1}{k+1} \text{ diverges. (it is shifted harmonic)}$$

$$x = -7 \quad \sum_{k=1}^{\infty} \frac{(-5)^k}{(k+1)5^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k+1} \text{ is an alternating series}$$

$$a_k = \frac{1}{k+1} < \frac{1}{k+2} = a_{k+1} \Rightarrow \{a_k\} \text{ is decreasing}$$

$$\lim_{k \rightarrow \infty} a_k = 0$$

By alternating series test (AST), $\sum_{k=1}^{\infty} \frac{(-1)^k}{k+1}$ converges, and it converges conditionally, since $\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{k+1} \right| = \sum_{k=1}^{\infty} \frac{1}{k+1}$ diverges

Therefore, the IOC is $[-7, 3)$

Q1 #1 2. (16 points) Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive real numbers.

- (a) (6 points) Assume $\lim_{k \rightarrow \infty} a_k = A$. Does $\{a_k + 8\}$ converge? If so, what is the value? If not, give a brief explanation why.

Answer:

$$\lim_{k \rightarrow \infty} (a_k + 8) = (\lim_{k \rightarrow \infty} a_k) + 8 = A + 8 \text{ is finite}$$

$\{a_k + 8\}$ converges

Note: there is only one way to check the convergence of a sequence.

$\{a_k\}$ converges iff $\lim_{k \rightarrow \infty} a_k$ exists

- (b) (5 points) Assume $\sum_{k=1}^{\infty} a_k = 15$. Does $\sum_{k=1}^{\infty} (a_k + 1)$ converge? If so, what is the value? If not, give a brief explanation why.

Answer:

Since $\sum_{k=1}^{\infty} a_k = 15$ converges, $\lim_{k \rightarrow \infty} a_k = 0$

$$\lim_{k \rightarrow \infty} (a_k + 1) = (\lim_{k \rightarrow \infty} a_k) + 1 = 1 \neq 0$$

By divergence test, $\sum_{k=1}^{\infty} (a_k + 1)$ diverges.

- △ (c) (5 points) Suppose $\sum_{k=1}^{\infty} a_k$ converges. Show that $\sum_{k=1}^{\infty} \frac{ka_k}{2k+3}$ converges or show that it diverges.

Answer: $\frac{k a_k}{2k+3} = \frac{a_k}{2 + \frac{3}{k}} < \frac{a_k}{2} \text{ for } k \geq 1$

Since $\sum_{k=1}^{\infty} a_k$ converges, $\sum_{k=1}^{\infty} \frac{ka_k}{2k+3}$ also converges by comparison test.

Or

$$\lim_{k \rightarrow \infty} \frac{\frac{k a_k}{2k+3}}{a_k} = \lim_{k \rightarrow \infty} \frac{k}{2k+3} = \frac{1}{2}$$

$\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} \frac{ka_k}{2k+3}$ both converge by limit comparison test.

Sep. 7 3. (20 points) Consider the function $f(x) = \sin(x^4)$.

Recitation Notes (a) (8 points) Find the MacLaurin series for f .

$$\int_0^{0.3} e^{-x^2} dx \quad \text{Answer: } \sin(0) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Chapter 10 Review

55

$$\sin(x^4) = \sum_{k=0}^{\infty} \frac{(-1)^k (x^4)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{8k+4}}{(2k+1)!}$$

(b) (8 points) Use your work in part (a) to write an infinite series which would estimate the value of

$$\int_0^1 \sin(x^4) dx.$$

$$\text{Answer: } \int_0^1 \sin(x^4) dx$$

$$= \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{8k+4}}{(2k+1)!} dx$$

$$= \sum_{k=0}^{\infty} \int_0^1 \frac{(-1)^k x^{8k+4}}{(2k+1)!} dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left. \frac{x^{8k+5}}{8k+5} \right|_0^1$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 1}{(2k+1)! (8k+5)}$$

(c) (4 points) Write an expression which would estimate the error in using n terms to estimate the value of the integral in part (b).

Answer: The series is an alternating series

$$|R_n(x)| \leq |(n+1)^{\text{st}} \text{ term}| = \frac{1}{(2n+1)! (8n+5)}$$

Aug. 29 4. (20 points) Determine whether the following series converge or diverge.

Recitation Notes (a) (10 points) $\sum_{k=1}^{\infty} \ln\left(\frac{k+3}{k+2}\right)$

Answer:

$$\ln\left(\frac{k+3}{k+2}\right) = \ln(k+3) - \ln(k+2)$$

$$\begin{aligned} S_n &= \cancel{\ln 4} - \ln 3 + \cancel{\ln 5} - \cancel{\ln 4} + \dots + \ln(n+3) - \cancel{\ln(n+2)} \\ &= \ln(n+3) - \ln 3 \end{aligned}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \ln(n+3) - \ln 3 = \infty$$

The series diverges.

Aug. 29 Recitation Notes (b) (10 points) $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$

Answer: Let $f(x) = \frac{1}{x(\ln x)^2}$

$f(x)$ is positive, continuous and decreasing for $x > 1$

$$\int_2^b \frac{1}{x(\ln x)^2} dx = \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x(\ln x)^2} dx$$

$$\text{Let } y = \ln x \quad dy = \frac{1}{x} dx$$

$$\begin{aligned} \int \frac{1}{x(\ln x)^2} dx &= \int \frac{1}{y^2} dy = \int y^{-2} dy \\ &= -y^{-1} + C = -\frac{1}{\ln x} + C \end{aligned}$$

$$\lim_{b \rightarrow \infty} \int_2^b \frac{1}{x(\ln x)^2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{\ln x} \right]_2^b = \frac{1}{\ln 2}$$

Since $\int_2^b \frac{1}{x(\ln x)^2} dx$ converges, $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges by integral test.

Sep 5 5. (15 points) Consider $g(x) = \ln(1 + 3x)$, which you will use to estimate $\ln(0.7)$.

Relativity Notes (a) (2 points) Where you should you center a polynomial approximation for g , and why?

$\sqrt{1.06}$

Chapter 10 Review

#57

Answer: center $a = 0$

Because $\ln 1 = 0$ and all derivatives of $g(x)$ are easy to evaluate at $a = 0$

(b) (9 points) Write a quadratic polynomial $p_2(x)$ which will approximate g near your chosen center.

$$\text{Answer: } P_2(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2$$

$$g(0) = \ln 1 = 0$$

$$g'(x) = \frac{3}{1+3x} \quad g'(0) = 3$$

$$g''(x) = -\frac{9}{(1+3x)^2} \quad g''(0) = -9$$

$$P_2(x) = 3x - \frac{9}{2}x^2$$

(c) (4 points) Estimate the absolute error in using p_2 in place of g to approximate $\ln(0.7)$. Your answer should be a numerical value, but need not be simplified.

$$\text{Answer: } R_2(x) = \frac{g^{(3)}(c)}{3!}x^3 \quad g^{(3)}(c) = \frac{54}{(1+3c)^3} \quad c \text{ is between } 0 \text{ and } x$$

$$\ln(0.7) = \ln(1-0.3) = \ln[1 + 0.3 \cdot (-0.1)]$$

$$\ln(0.7) \approx P_2(-0.1)$$

$$|R_2(-0.1)| = \left| \frac{g^{(3)}(c)}{3!} (-0.1)^3 \right| = \left| \frac{(0.1)^3 \cdot 54}{6 \cdot (1+3c)^3} \right| \leq \frac{(0.1)^3 \cdot 54}{6 \cdot (0.7)^3} = \frac{9}{343}$$

c is between 0 and -0.1

6. (15 points) For each of the following statements, circle T if the statement is true and F if the statement is false. You do not need to show any explanations for this problem.

(a) (3 points) T / F

Every sequence either converges or diverges to $\pm\infty$.

$$\lim_{n \rightarrow \infty} (-1)^n \text{ DNE}$$

(b) (3 points) T / F

$$\sum_{k=2}^{\infty} \frac{3}{5^k} = \frac{3}{20}.$$

$$= \sum_{k=2}^{\infty} 3 \cdot \left(\frac{1}{5}\right)^k$$

$$a = 3 \cdot \left(\frac{1}{5}\right)^2 \text{ and } r = \frac{1}{5} \quad \sum_{k=2}^{\infty} \frac{3}{5^k} = \frac{3 \left(\frac{1}{5}\right)^2}{1 - \frac{1}{5}} = \frac{3}{20}$$

(c) (3 points) T / F

Suppose $\sum_{k=0}^{\infty} a_k$ is a series whose sequence of partial sums is given by $S_n = \frac{1}{\sqrt{n}}$. In this case, the series $\sum a_k$ converges.

$\sum_{k=0}^{\infty} a_k$ converges iff $\lim_{n \rightarrow \infty} S_n$ exists

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0, \text{ so } \sum_{k=0}^{\infty} a_k \text{ converges}$$

(d) (3 points) T / F

If $\lim_{k \rightarrow \infty} a_k = 0$, $\sum_{k=0}^{\infty} a_k$ could converge.

Example: $\sum_{k=1}^{\infty} \frac{1}{n^2}$ converges

△ (e) (3 points) T / F

If $f(x) = \sum_{k=1}^{\infty} (-1)^k x^{2k+3}$, then $f^{(18)}(0) = 0$.

$$f(x) = \sum_{k=0}^{\infty} c_k x^k \text{ and } c_k = \frac{f^{(k)}(0)}{k!}$$

To find $f^{(18)}(0)$, we only need to exam the coefficient of x^{18}

Since all powers are odd ($2k+3$ is odd for all k), $f^{(18)}(0) = 0$

