

§ 15.8

Recall Green's thm $\oint_C \vec{F} \cdot \vec{n} ds = \iint_R \operatorname{div}(\vec{F}) dA$

2D flux double integral

Divergence thm:

Conditions: R is a 3D region which is connected and simply connected. The boundary of R is an oriented surface S . \vec{n} is outward unit normal vector of S .

$\vec{F} = \langle f, g, h \rangle$ f, g, h have continuous first derivatives on R

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_R \nabla \cdot \vec{F} dV$$

3D flux triple integral

Note: D is a 3D hollow region bdd by S_1 and S_2 , D is simply connected.

And Divergence thm can be applied on D .

e.g. D is the region bdd by $r=2$ and $r=\sqrt{5}$ two spheres.

Chapter 15

Review #65

(D is simply connected)

$$\vec{F} = 2\langle x, y, z \rangle \cdot \sqrt{x^2+y^2+z^2}$$

similar

calculate $\iint_S \vec{F} \cdot \vec{n} ds$

HW12

36

Answer: By divergence thm $\iint_S \vec{F} \cdot \vec{n} ds = \iiint_D \nabla \cdot \vec{F} dV$

$$\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 2\sqrt{x^2+y^2+z^2} + \frac{2x^2}{\sqrt{x^2+y^2+z^2}} + 2\sqrt{x^2+y^2+z^2} + \frac{2y^2}{\sqrt{x^2+y^2+z^2}} + 2\sqrt{x^2+y^2+z^2}$$
$$+ \frac{2z^2}{\sqrt{x^2+y^2+z^2}} = 8\sqrt{x^2+y^2+z^2}$$

$$\iiint_D \nabla \cdot \vec{F} dV = \iiint_D 8\sqrt{x^2+y^2+z^2} dV$$

$$= \int_0^{2\pi} \int_0^\pi \int_2^5 8\rho \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 72\pi$$

#

Example from old final (2016 Spring)

Compute the net outward flux of $\vec{F} = \left\langle \frac{x^3}{3}, \frac{y^3}{3}, \frac{z^3}{3} \right\rangle$ across the boundary of the region within the sphere $x^2+y^2+z^2=16$ and above $z=0$ plane

Answer: $\iint_S \vec{F} \cdot \vec{n} ds = \iiint_R \nabla \cdot \vec{F} dv$

$$\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 2(x^2 + y^2 + z^2)$$

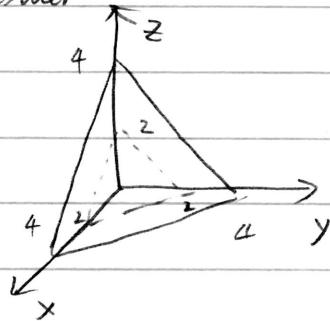
$$\begin{aligned} \iiint_R \nabla \cdot \vec{F} dv &= \int_0^{\pi} \int_0^{\frac{\pi}{2}} \int_0^4 (2r^2) r^2 \sin \varphi dr d\varphi d\theta \\ &= \frac{2048}{5} \pi \end{aligned}$$

Example #26 S15.8 (HW12 #37)

$\vec{F} = \langle x^2, -y^2, z^2 \rangle$. D is the region bdd by $z = 4 - x - y$, $z = 2 - x - y$ in 1st octant

Q: Find net out flux.

Answer



$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_D \nabla \cdot \vec{F} dv$$

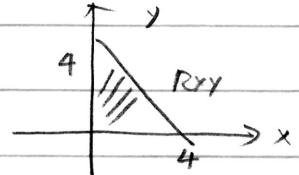
$$= \iiint_{D_1} \nabla \cdot \vec{F} dv - \iiint_{D_2} \nabla \cdot \vec{F} dv$$

D_1 : bigger tetrahedron

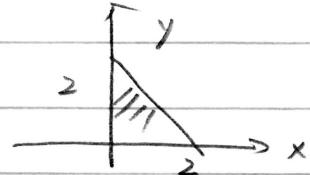
D_2 : smaller tetrahedron

$$\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 2x - 2y - 2z$$

$$\begin{aligned} ① \quad \iiint_{D_1} 2x - 2y - 2z dv &= \iint_{R_{xy}} \int_0^{4-x-y} 2x - 2y - 2z dz dA \\ &= \int_0^4 \int_0^{4-x} \int_0^{4-x-y} 2x - 2y - 2z dz dy dx \end{aligned}$$



$$\begin{aligned} ② \quad \iiint_{D_2} 2x - 2y - 2z dv &= \iint_{R_{xy}} \int_0^{2-x-y} 2x - 2y - 2z dz dA \\ &= \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 2x - 2y - 2z dz dy dx \end{aligned}$$



$$\iint_S \vec{F} \cdot \vec{n} ds = ① - ②$$

#

Chapter 15 Review

Tools:

- Line integral: Parametric equation of curve C is $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

<1> Scalar function. $\int_C f ds = \int_a^b f(t) \cdot |\vec{r}'(t)| dt$

<2> Vector field $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(t) \cdot \vec{r}'(t) dt$

Parametric equations of common curves

- ① Line segment from (a_1, b_1, c_1) to (a_2, b_2, c_2)

$$\vec{r}(t) = \langle a_1, b_1, c_1 \rangle + t \cdot \langle a_2 - a_1, b_2 - b_1, c_2 - c_1 \rangle \quad 0 \leq t \leq 1$$

- ② 2D circle $(x-a)^2 + (y-b)^2 = r^2$ counterclockwise

$$\vec{r}(t) = \langle a + r \cos t, b + r \sin t \rangle \quad 0 \leq t \leq 2\pi$$

- ③ 2D ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ counterclockwise

$$\vec{r}(t) = \langle a \cos t, b \sin t \rangle \quad 0 \leq t \leq 2\pi$$

- ④ General curve $y = f(x) \quad a \leq x \leq b$

$$\vec{r}(t) = \langle t, f(t) \rangle \quad a \leq t \leq b$$

- Surface integral

Using parametric form of surface $\vec{r}(u, v), (u, v) \in R$

<1> Scalar function $\iint_S f ds = \iint_R f(u, v) \cdot |\vec{r}_u \times \vec{r}_v| dA$

<2> Vector field: $\iint_S \vec{F} \cdot \vec{n} ds = \iint_R \vec{F}(u, v) \cdot (\pm \vec{r}_u \times \vec{r}_v) dA$

(Special) Using explicit form of surface $z = z(x, y) \quad (x, y) \in R$

<1> Scalar function: $\iint_S f ds = \iint_R f(x, y) \sqrt{1 + z_x^2 + z_y^2} dA$

<2> Vector field $\iint_S \vec{F} \cdot \vec{n} ds = \iint_R \vec{F} \cdot (\pm \langle -z_x, -z_y, 1 \rangle) dA$

Parametric forms of some surface \rightarrow Textbook Page

Note: Arc length = $\int_C 1 ds$

Surface area = $\iint_S 1 ds$

- Definitions: $\vec{F} = \langle f, g, h \rangle$

① Divergence $\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$

$$\textcircled{2) } \operatorname{curl} \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

Three theorems

Circulation.

$$2D \quad \text{Green's thm} \quad \oint_C \vec{F} \cdot d\vec{r} = \iint_R 2\operatorname{curl} F \, dA = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

$$3D \quad \text{Stokes' thm} \quad \oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, ds$$

Flux

$$2D \quad \text{Green's thm} \quad \oint_C \vec{F} \cdot \vec{n} \, ds = \iint_R \operatorname{div}(\vec{F}) \, dA = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dA$$

$$3D \quad \text{Divergence thm} \quad \iint_S \vec{F} \cdot \vec{n} \, ds = \iiint_R \nabla \cdot \vec{F} \, dv$$

Note: Be sure to review the conditions for three thms.

Note: "simply connected"

2D: no hole 0 ✗

3D: hollow region is simply connected ✓

* Conservative vector field

① \vec{F} is conservative, i.e. $\vec{F} = \nabla \varphi$.

② $\oint_C \vec{F} \cdot d\vec{r} = \varphi(B) - \varphi(A)$ path independent

③ $\oint_C \vec{F} \cdot d\vec{r} = 0$

④ $\nabla \times \vec{F} = \vec{0}$

Classic problem: given $\vec{F} = \langle f, g, h \rangle$, find φ

Step 1: $f = \varphi_x \Rightarrow \boxed{\varphi = \int f \, dx + C(y, z)}$

Step 2: $\varphi_y = [\int f \, dx + C(y, z)]_y = g \quad \text{and solve for } C(y, z)$

Then $\varphi = \int f \, dx + \boxed{\quad} + d(z)$

Step 3 $\varphi_z = [\int f \, dx + \boxed{\quad} + d(z)]_z = h \quad \text{and solve for } d(z)$

Then update φ

Note This procedure works for all problems