

Aug. 22

§ 9.1 Overview of sequence and series

Part I. Sequence

e.g. $\{2, -4, 6, -8, 10, -12, \dots\}$

Def: A sequence $\{a_n\}_{n=1}^{\infty}$ is an ordered list of numbers of the form $\{a_1, a_2, a_3, \dots\}$

a_n is called a term and n is called an index

$\{a_n\}$ may be generated by

① Explicit formula, i.e. $a_n = f(n)$ for $n=1, 2, 3, \dots$

② Recurrence relation, i.e. $a_{n+1} = f(a_n)$ $n=1, 2, \dots$

For the example provided above

① $a_n = (-1)^n \cdot 2n \quad n=1, 2, 3, \dots$

② $a_n = (-1)^n \cdot 2n \quad a_{n+1} = (-1)^{n+2} \cdot 2(n+1) \quad \frac{a_{n+1}}{a_n} = \frac{(-1)^{n+2} \cdot 2(n+1)}{(-1)^n \cdot 2n} = -\frac{n+1}{n} \Rightarrow a_{n+1} = -\frac{n+1}{n} a_n$

Two types of problems

Type I. Given explicit or recurrence formula, write out the first few terms.

Examples:

(a) $a_n = \frac{(-1)^n \cdot n}{n^2 + 1}$

$$n=1 \quad a_1 = \frac{(-1)^1 \cdot 1}{1^2 + 1} = -\frac{1}{2}$$

$$n=2 \quad a_2 = \frac{(-1)^2 \cdot 2}{2^2 + 1} = \frac{2}{5}$$

$$n=3 \quad a_3 = \frac{(-1)^3 \cdot 3}{3^2 + 1} = -\frac{3}{10}$$

(b) $a_{n+1} = 2a_n + 1$ and $a_1 = 1$

$$a_2 = 2a_1 + 1 = 2 \cdot 1 + 1 = 3$$

$$a_3 = 2a_2 + 1 = 2 \cdot 3 + 1 = 7$$

$$a_4 = 2a_3 + 1 = 2 \cdot 7 + 1 = 15$$

Type II: Given the first few terms of a sequence, find the explicit or recurrence formula

Examples:

(a) $\{a_n\} = \{-2, 5, 12, 19, \dots\}$

Recurrence formula: $a_{n+1} = a_n + 7$

Explicit formula: $a_1 = -2 \quad a_2 = -2 + 7 \quad a_3 = a_2 + 7 = -2 + 7 + 7 = -2 + 7 \times 2$

$$\Rightarrow a_n = -2 + (n-1) \cdot 7 \quad n=1, 2, 3, \dots$$

or $a_n = -2 + n \cdot 7 \quad n=0, 1, 2, 3, \dots$

$$(b) \{b_n\} = \{3, 6, 12, 24, 48, \dots\}$$

Recurrence relation: $a_{n+1} = 2a_n$

$$\text{Explicit formula: } a_1 = 3 \quad a_2 = 2 \cdot a_1 = 2 \cdot 3 \quad a_3 = 2 \cdot a_2 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$$

$$a_4 = 2 \cdot a_3 = 2^3 \cdot 3$$

$$\Rightarrow a_n = 2^n \cdot 3 \quad n=1, 2, 3 \dots$$

$$\text{or } a_n = 2^n \cdot 3 \quad n=0, 1, 2, 3 \dots$$

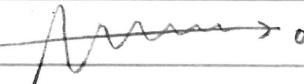
Def: If $\{a_n\}$ approach a unique number L as n increases, i.e. $\lim_{n \rightarrow \infty} a_n = L$, then the sequence converges to L . If $\{a_n\}$ does not converge, it diverges.

Examples: Find the limits if the sequences converge, explain why if they diverge

$$(a) \left\{ \frac{(-1)^n}{n^2 + 1} \right\}_{n=1}^{\infty}$$

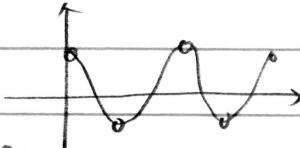
$$\left\{ -\frac{1}{2}, \frac{1}{5}, -\frac{1}{10}, \frac{1}{17}, \dots \right\}$$

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n^2 + 1} = 0 \quad \text{it converges.}$$



$$(b) \left\{ \cos(n\pi) \right\}_{n=1}^{\infty}$$

$$\left\{ -1, 1, -1, 1, \dots \right\}$$



It diverges, since $\lim_{n \rightarrow \infty} \cos(n\pi)$ DNE

$$(c) \left\{ a_n \right\}_{n=1}^{\infty} \quad a_{n+1} = -2a_n \quad a_1 = 1$$

$$a_1 = 1 \quad a_2 = -2 \quad a_3 = 4, \quad a_4 = -8 \dots$$

It diverges, since $\lim_{n \rightarrow \infty} a_n$ DNE



$$(d) \left\{ \frac{4n^3}{n^3 + 1} \right\}_{n=1}^{\infty}$$

$$\lim_{n \rightarrow \infty} \frac{4n^3}{n^3 + 1} = \lim_{n \rightarrow \infty} \frac{4}{1 + \frac{1}{n^3}} = 4$$

Part II Series

Def: Given a sequence $\{a_1, a_2, \dots\}$, the sum $a_1 + a_2 + a_3 + \dots = \sum_{n=1}^{\infty} a_n$ is called an infinite series. The partial sum S_n is defined as $S_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^n a_k$. The partial sums $\{S_1, S_2, S_3, \dots\}$ form a sequence.

If $\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k = L$, then $\sum_{k=1}^{\infty} a_k$ converges.

Examples: Convergence of a series

(a) $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ telescoping series

Find S_n and check $\lim_{n \rightarrow \infty} S_n$

$$\frac{1}{k(k+1)} = \frac{a}{k} + \frac{b}{k+1} \quad a(k+1) + b \cdot k = 1 \Rightarrow a = -b \quad a = 1$$

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$S_1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$S_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3} = \frac{2}{3}$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$S_n = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} \dots + \cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

$\lim_{n \rightarrow \infty} S_n = 1 \Rightarrow$ the series converges

Practice. $\sum_{k=1}^{\infty} \frac{2}{(2k-1)(2k+1)}$ & $\sum_{k=1}^{\infty} \frac{1}{4k^2-1}$

(b) $\sum_{k=1}^{\infty} (-1)^k \cdot k$

Find S_n and check $\lim_{n \rightarrow \infty} S_n$

$$S_1 = (-1)^1 \cdot 1 = -1$$

$$S_2 = -1 + (-1)^2 \cdot 2 = -1 + 2 = 1$$

$$S_3 = S_2 + a_3 = 1 + (-1)^3 \cdot 3 = 1 - 3 = -2$$

$$S_4 = S_3 + a_4 = -2 + (-1)^4 \cdot 4 = 2$$

$\lim_{n \rightarrow \infty} S_n$ DNE

Summary:

Sequence

Series

Def.

$$\{a_n\}_{n=1}^{\infty}$$

$$\sum_{n=1}^{\infty} a_n$$

Convergence

$$\lim_{n \rightarrow \infty} a_n = L$$

converges

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k, a_k = L$$

converges

Correspondences between sequence/series and functions

	Sequence/series	function
Independent variable	n	x
Dependent variable	a_n	$f(x)$
Domain	$n=1, 2, 3, \dots$	$x \in \mathbb{R}$
Accumulation	$S_n = \sum_{k=1}^n a_k$ Finite	$\int_1^n f(x) dx$
Infinite	$\sum_{k=1}^{\infty} a_k$	$\int_1^{\infty} f(x) dx$