Inverse Euler Formula

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$A\cos(\omega_0 t + \phi) = A\left(\frac{e^{j(\omega_0 t + \phi)} + e^{-j(\omega_0 t + \phi)}}{2}\right)$$

$$= \frac{1}{2}\left(X e^{j\omega_0 t} + X^* e^{-j\omega_0 t}\right)$$

$$A\cos(\omega_0 t + \phi_0) \longleftrightarrow \overbrace{A e^{j\phi_0}}^{X} \qquad A>0$$

phasor

$$A\cos(\omega_0 t + \phi_0) = \Re e\{X e^{j\omega_0 t}\} \qquad A>0$$

phasor

$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = \overset{?}{A}\cos(\omega_0 t + \overset{?}{\phi})$$

$$\sum_{k=1}^{N} A_k \Re e\{e^{j\omega_0 t} e^{j\phi_k}\} = A\cos(\omega_0 t + \phi)$$

$$\Re e\left\{\left[\sum_{k=1}^{N} A_k e^{j\phi_k}\right] e^{j\omega_0 t}\right\} = A\cos(\omega_0 t + \phi)$$

$$\Re e\{A e^{j\phi} e^{j\omega_0 t}\} = A\cos(\omega_0 t + \phi)$$

$$\sum_k A_k e^{j\phi_k} = A e^{j\phi}$$

Example

$$x_1(t) = 1.7\cos(20\pi t + 70\pi/180)$$

$$x_2(t) = 1.9\cos(20\pi t + 200\pi/180)$$

$$x_1(t) + x_2(t)\ ?$$

convert to rectangular

$$X_1 = 1.7e^{j70\pi/180} = 0.5814 + j1.597$$

$$X_2 = 1.9e^{j200\pi/180} = -1.785 - j0.6498$$

$$X_1 + X_2 = (0.5814 + j1.597) + (-1.785 - j0.6498)$$

$$= -1.204 + j0.9476$$

convert to polar

$$= 1.532e^{j141.79\pi/180}$$

$\therefore$

$$x_1(t) + x_2(t) = 1.532\cos(20\pi t + 141.79\pi/180)$$

$$= 1.532\cos(20\pi(t + 0.0394))$$

CHAPTER 3
Spectrum Representation

dc value

$$x(t) = \overbrace{A_0}^{\text{dc value}} + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$$

additive linear combination

$$x(t) = \underset{\underset{A_0}{\parallel}}{X_0} + \sum_{k=1}^{N} \mathcal{R}e\left\{ X_k e^{j2\pi f_k t} \right\}$$

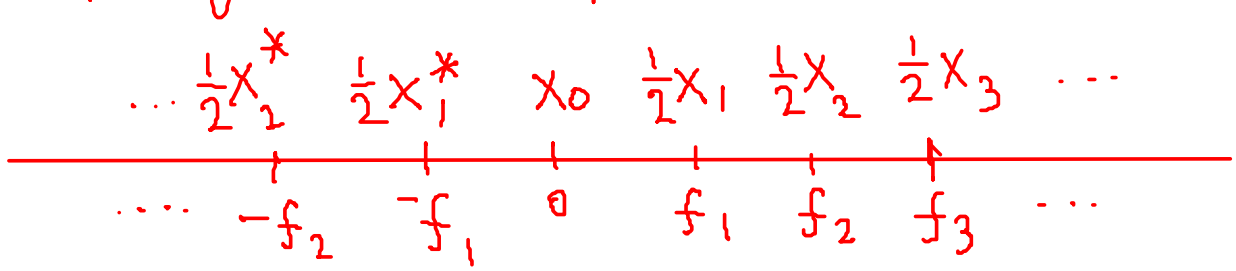
$$X_k = A_k e^{j\phi_k}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k^*}{2} e^{-j2\pi f_k t} \right\}$$

two sided spectrum of $x(t)$

$$\left[(0, X_0), \left(f_1, \tfrac{1}{2}X_1\right), \left(-f_1, \tfrac{1}{2}X_1^*\right), \left(f_2, \tfrac{1}{2}X_2\right), \left(-f_2, \tfrac{1}{2}X_2^*\right), \cdots \right]$$

Frequency domain representation



Example

$$x(t) = 10 + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2)$$

$$x(t) = 10 + 7e^{-j\frac{\pi}{3}} e^{j200\pi t} + 7e^{j\frac{\pi}{3}} e^{-j200\pi t} + 4e^{j\frac{\pi}{2}} e^{j500\pi t} + 4e^{-j\frac{\pi}{2}} e^{-j500\pi t}$$

spectrum

$$\left[(0, 10),\ (100, 7e^{-j\frac{\pi}{3}}),\ (-100, 7e^{j\frac{\pi}{3}}),\ (250, 4e^{j\frac{\pi}{2}}),\ (-250, 4e^{-j\frac{\pi}{2}}) \right]$$

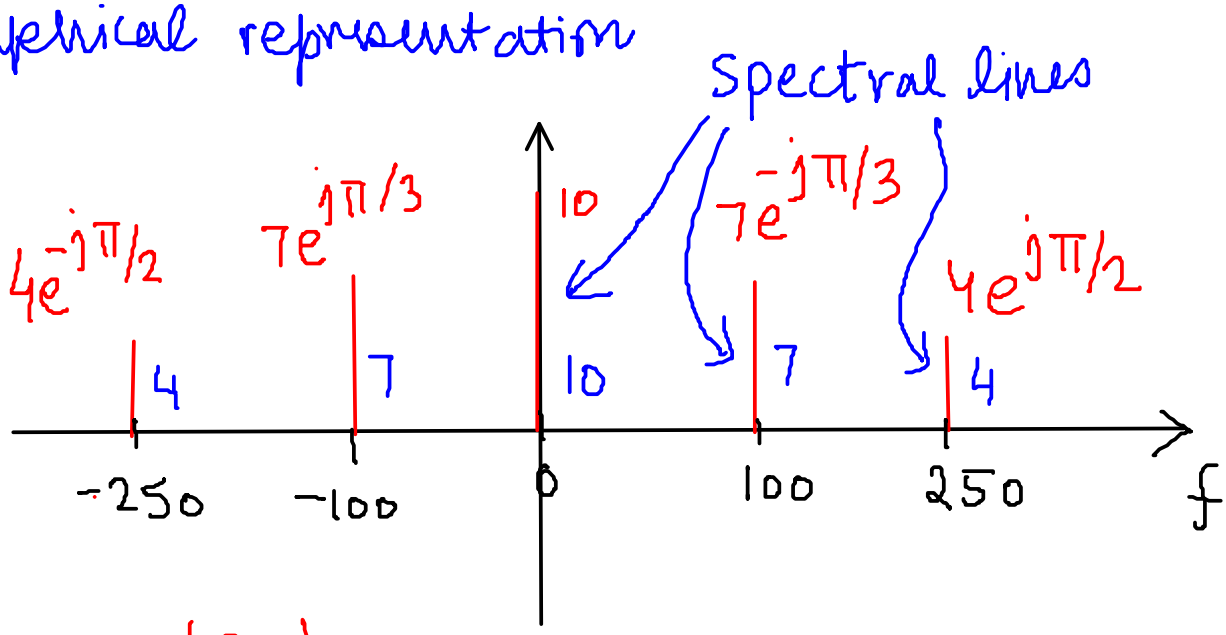
Notation change

$$a_k = \begin{cases} A_0 & \text{if } k=0 \\ \frac{1}{2}A_k e^{j\phi_k} & \text{if } k \neq 0 \end{cases}$$

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k}$$

(Note: $f_0 = 0 \rightarrow dc$)

Graphical representation



plot of $|a_k|$ vs. f

$$|a_k| = \begin{cases} A_0 & \text{if } k=0 \\ \frac{1}{2}A_k & \text{if } k \neq 0 \end{cases}$$

if $x(t)$ is real then $a_{-k} = a_k^*$

$$|a_k| = |a_{-k}|$$

$$\phi_k = -\phi_{-k}$$

Multiplication of Sinusoids

$\frac{1}{2}$ Hz 5 Hz

$$x(t) = \cos(\pi t)\sin(10\pi t)$$

$$x(t) = \left(\frac{e^{j\pi t} + e^{-j\pi t}}{2}\right)\left(\frac{e^{j10\pi t} - e^{-j10\pi t}}{2j}\right)$$

$$x(t) = \frac{1}{4j}\left(e^{j\pi t} + e^{-j\pi t}\right)\left(e^{j10\pi t} - e^{-j10\pi t}\right)$$

$$= \frac{1}{4j}e^{j11\pi t} + \frac{1}{4j}e^{j9\pi t} - \frac{1}{4j}e^{-j9\pi t} - \frac{1}{4j}e^{-j11\pi t}$$

$$\frac{1}{j} = -j$$

$$= \frac{1}{4j} e^{j11\pi t} + \frac{1}{4j} e^{j9\pi t} + \frac{j}{4} e^{-j9\pi t} + \frac{j}{4} e^{-j11\pi t}$$

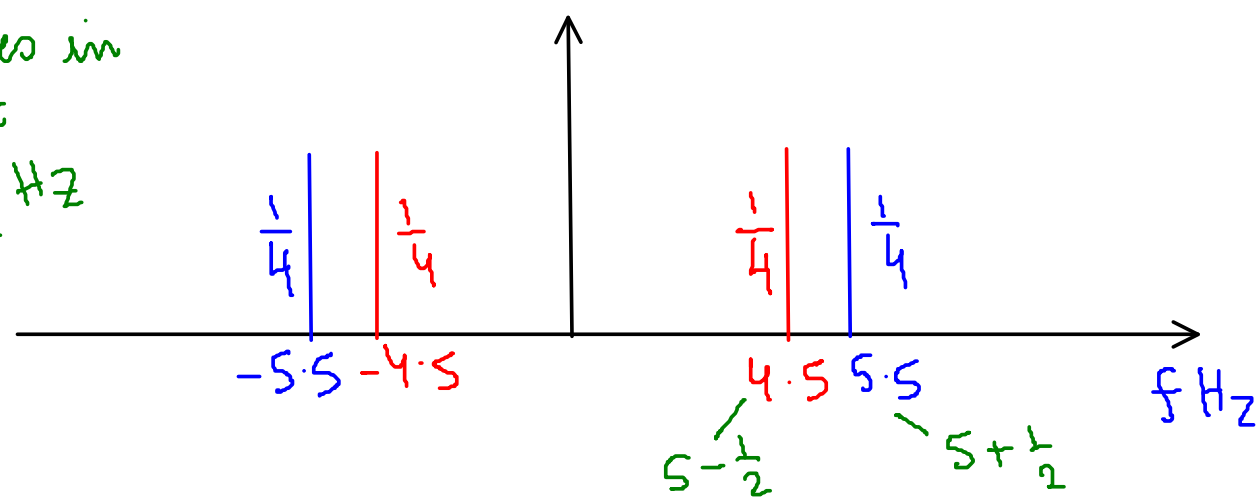
$$j = e^{j\frac{\pi}{2}} \qquad \frac{1}{j} = e^{-j\frac{\pi}{2}}$$

$$= \frac{1}{4} e^{-j\frac{\pi}{2}} e^{j11\pi t} + \frac{1}{4} e^{-j\frac{\pi}{2}} e^{j9\pi t} + \frac{1}{4} e^{j\frac{\pi}{2}} e^{-j9\pi t} + \frac{1}{4} e^{j\frac{\pi}{2}} e^{-j11\pi t}$$

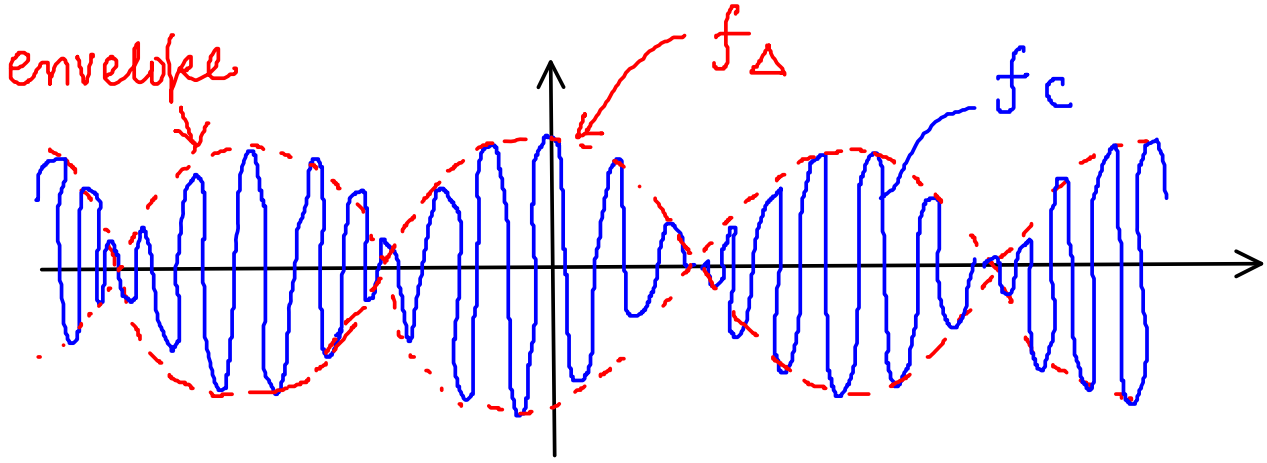
$$= \frac{1}{2}\cos(\underbrace{11\pi}_{5.5\,Hz} t - \pi/2) + \frac{1}{2}\cos(\underbrace{9\pi}_{4.5\,Hz} t - \pi/2)$$

frequencies in product

5 Hz, $\frac{1}{2}$ Hz



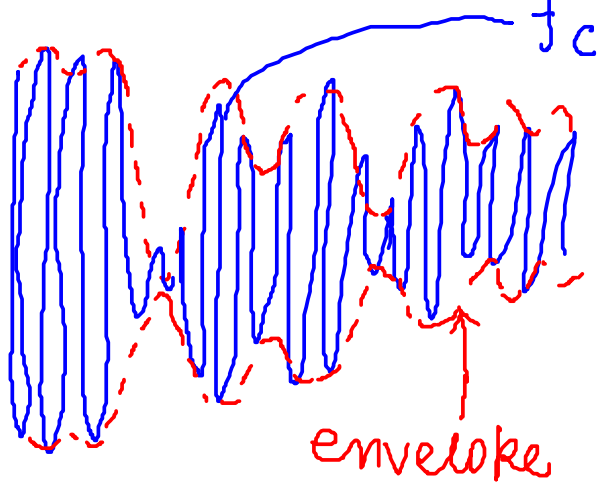
$$x(t) = \cos(2\pi f_\Delta t)\cos(2\pi f_c t) \qquad f_\Delta \ll f_c$$



Beats

Amplitude Modulation (AM)

$f_c \rightarrow$ carrier frequency



AM radio transmission

Beat Notes

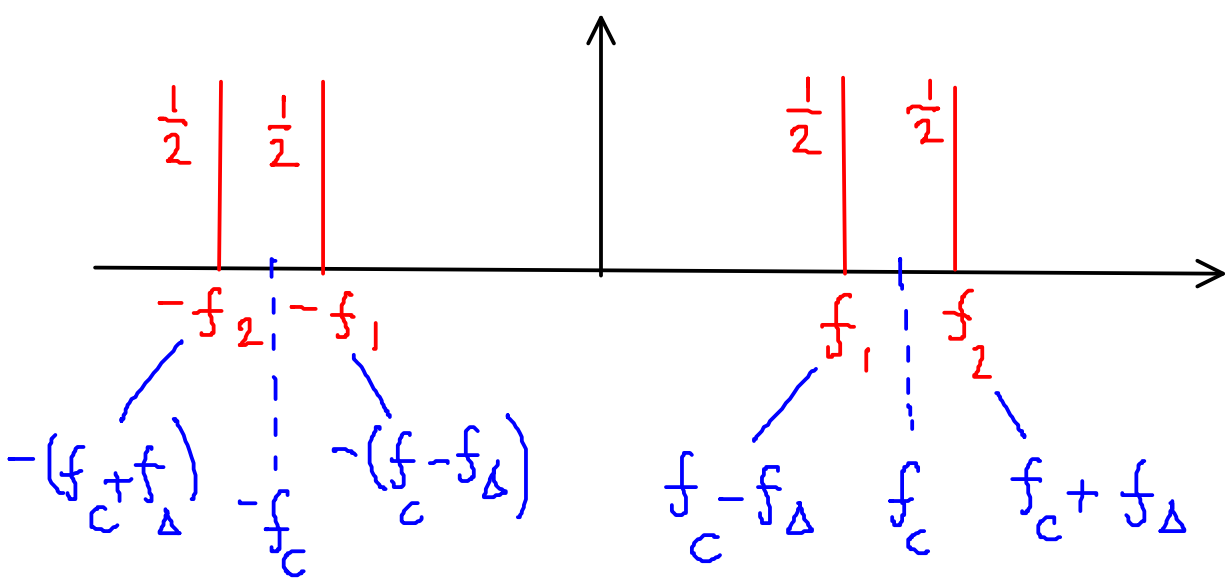
Beat notes produced by adding two sinusoids with nearly identical frequencies

$$x(t) = 2\cos(2\pi f_\Delta t)\cos(2\pi f_c t) \qquad f_\Delta \ll f_c$$

$$x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t)$$

$$f_1 = f_c - f_\Delta \qquad f_2 = f_c + f_\Delta \qquad f_1, f_2 \text{ close}$$

$$f_c = \frac{f_1 + f_2}{2} \qquad f_\Delta = \frac{f_2 - f_1}{2}$$



Amplitude Modulation

$$x(t) = v(t)\cos(2\pi f_c t)$$

example

$$v(t) = 5 + 4\cos(40\pi t)$$

$$f_\Delta = 20\,\text{Hz}$$

$$x(t) = [5 + 4\cos(40\pi t)]\cos(400\pi t)$$

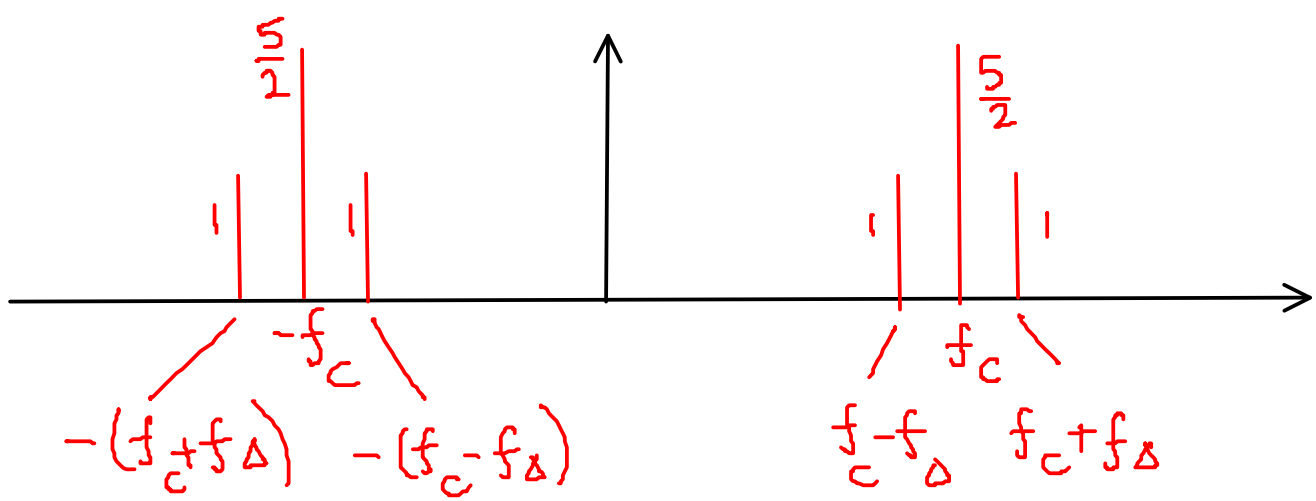
$$f_c = 200\,\text{Hz}$$

$$x(t) = 5\cos(400\pi t) + \underbrace{4\cos(40\pi t)\cos(400\pi t)}$$

$$200\,\text{Hz} \qquad f_c - f_\Delta \;,\; f_c + f_\Delta$$

$$180\,\text{Hz} \qquad 220\,\text{Hz}$$

Spectrum

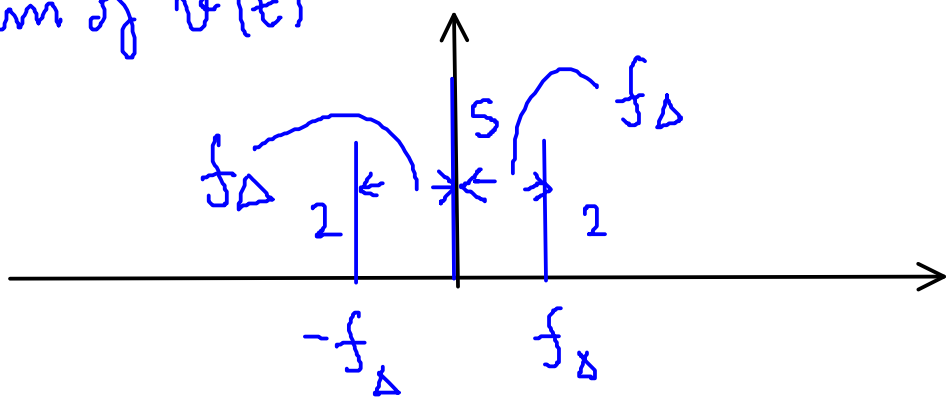
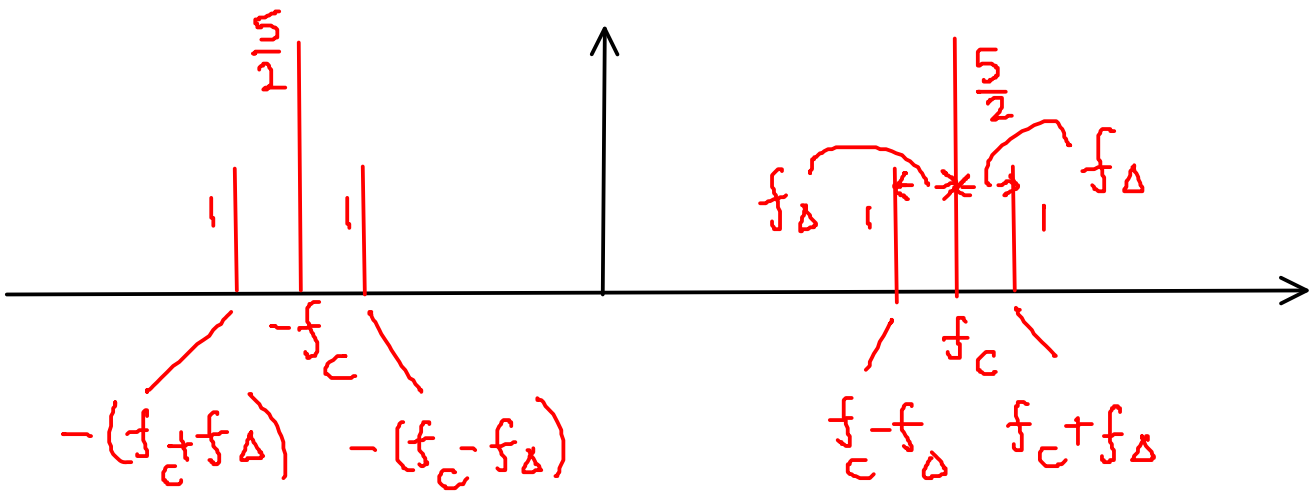


two sided spectrum of $v(t)$

$$v(t) = 5 + 4\cos(40\pi t)$$

$$f_\Delta = 20\,\text{Hz}$$

Spectrum of $v(t)$



Spectrum of $x(t) = v(t)\cos(2\pi f_c t)$