Aspects of the Monster Tower Construction: Geometric, Combinatorial, Mechanical, Enumerative

Lecture 3: Mechanical Aspects

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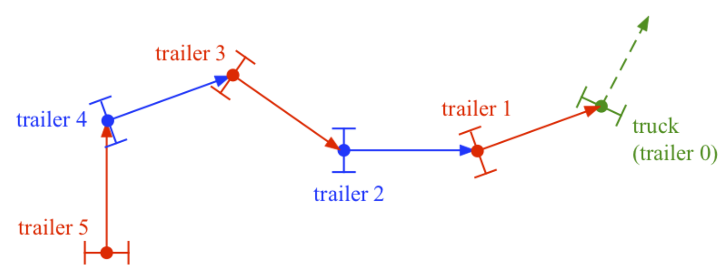
Table of Contents

- We are not as well acquainted with this aspect, so our reporting may be superficial or misguided. Nevertheless, we think that learning more about this connection may be helpful in both directions.
- You've just heard from an expert about the relevant dynamics, Alejandro Bravo-Doddoli. We may cover some of the same territory, while trying to tie it to the other aspects.

The configuration space for a truck with trailers

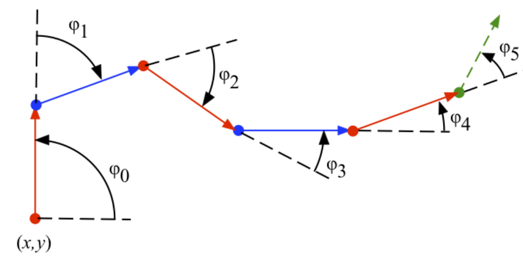
- We're looking at a *truck* pulling *n trailers* of unit length; we call the entire configuration a *train*.
  - We may say "trailer 0" instead of "truck."
- We idealize each trailer as a point together with a unit vector pointing in the direction of the previous trailer; the truck's unit vector may point in any direction.



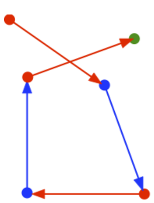
- The configuration space is $\mathbb{R}^2 \times (S^1)^{n+1}$, a manifold of dimension $n+3$.
- A point of the configuration space is specified by an $(n+3)$-tuple

$$(x, y, \varphi_0, \varphi_1, \ldots, \varphi_n).$$

- The first two coordinates (x, y) record the position of the last trailer (trailer n).
- The coordinates $\varphi_1, \ldots, \varphi_n$ record the *bending angles* formed by successive trailers, ordered from the back of the train to the front: φ_k records the angle formed at trailer $n-k$ between the unit vectors associated to trailers $n-k$ and $n-k+1$.
- φ_0 is the *heading angle* formed by the unit vector associated to the last trailer and the horizontal direction.



- Some unrealistic aspects:
  - The trailers can intersect.
  - We can have “accordion configurations," with trailers totally overlapping.



- In the following treatment we will assume that each of the angles is acute: $-\pi/2 \le \varphi_k \le \pi/2$.

- In Lecture 1, we noted that this configuration space is $\mathbb{R}^2_{\text{ray}}(n+1)$, the ray-monster space over the plane.
- The map

$$\mathbb{R}^2_{\text{ray}}(n+1) \to \mathbb{R}^2_{\text{ray}}(n)$$

just strips away the truck, and the first trailer then plays the role of the truck.

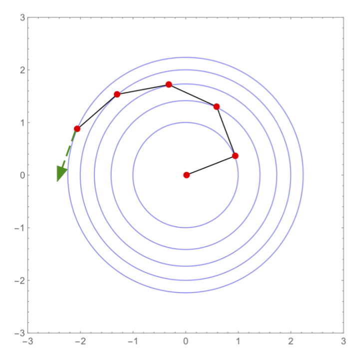
Driving the train; rigid trains

- The truck can be driven following any differentiable path, subject to the following condition:
  - The velocity vector must point in the direction that the truck is pointing (as indicated by its unit vector).
- If the velocity vector is zero, this is interpreted as a vacuous condition (automatically satisfied).
- It is also legal to drive the truck simply by turning, i.e., by keeping its position fixed while changing φ_n.

- The motion of each trailer is determined by two conditions:
  1. The distance between successive trailers must always be the unit length. (This is a holonomic constraint.)
  2. The velocity vector must point in the direction that the trailer is pointing (as specified by φ_i), i.e., the velocity vector must point in the direction of the previous trailer. (This is a nonholonomic constraint.)
- These constraints imply a certain system of differential equations, which, given the path of the truck, can be solved to determine the paths of the trailers.

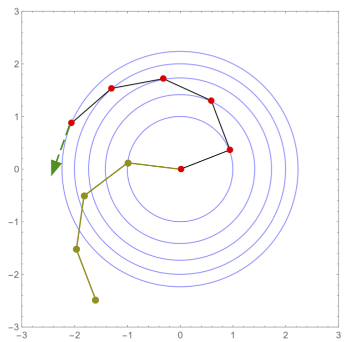
- Going in the other direction, if we are given the path of a trailer, then the process of obtaining the path of the previous trailer is a geometric process involving derivatives.
- Suppose we specify a differentiable path for the last trailer. At each point, draw the unit tangent vector, and then mark the head of this vector. The heads of all these vectors trace out the path of the prior trailer. Now repeat the procedure, moving forward in the train until you reach the truck.
- As an example, suppose you want to move the last trailer along a circle of radius r. Then the prior trailer should be moving along a circle of radius $\sqrt{r^2+1}$. Working forward, we obtain a sequence of concentric circles, with the truck driving along the outer circle.

- As a special case of this construction, take $r = 0$. Then the bending angles are $\varphi_k = \sin^{-1}\frac{1}{\sqrt{k}}$.



- Observe that the entire train will move rigidly, as if the angles between the successive trailers were welded at certain fixed angle. The last trailer isn't moving; it's just turning.

- To embellish this example, we can attach additional trailers to the end.



- Since the end isn't moving, we can satisfy the motion constraints by not moving these additional trailers at all. The overall motion then looks like this: the rear of the train isn't moving at all, while the front of the train is moving rigidly. Just one angle is changing.

Vector fields (infinitesimal motions)

- To probe the dynamics of the train, we will look at infinitesimal motions, i.e., vector fields on the configuration space.
- We want to distinguish two sorts of infinitesimal motions:
  - motions that simply preserve the distance between the trailers,
  - motions that correspond to legal ways of moving the train.
- The latter sort of motion is quite restrictive, since the only available infinitesimal motions of the truck are linear combinations of these two:
  - $v =$ infinitesimally turn the truck leftward
  - $f =$ infinitesimally drive the truck forward

- The first motion

$$v = \text{infinitesimally turn the truck leftward}$$

extends to a legal motion v_n of the train in a trivial way: the trailers don't move at all. In coordinates:

$$v_n = \frac{\partial}{\partial \varphi_n}.$$

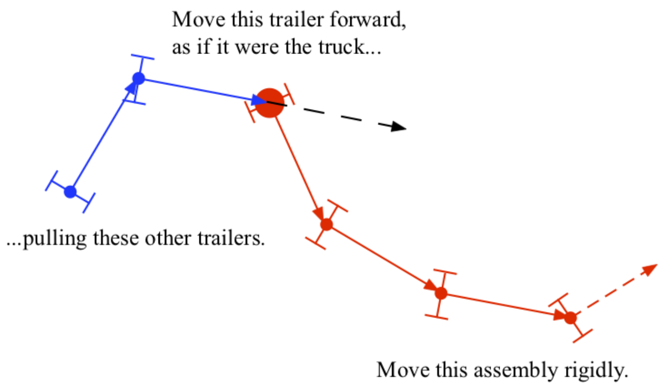
- The second motion f likewise extends to a motion f_n of the entire train. If we drive the truck forward, however, then we certainly expect the trailers to move. Thus the coordinate expression for f_n is more complicated.

- When we regard the configuration space as the ray-monster $\mathbb{R}^2_{\text{ray}}(n+1)$, the vector field v_n is a vertical vector. Physically, this means that when we forget about the truck, the remainder of the train is stationary.
- Taking both v_n and f_n, they span a rank 2 subbundle of the tangent bundle of the configuration space. In fact it's the focal bundle.
- Jean introduces two sorts of vector fields

$$f_k \quad \text{and} \quad v_k \qquad (0 \le k \le n)$$

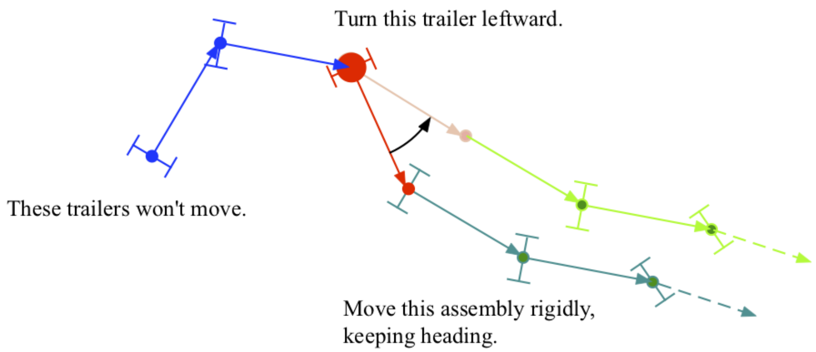
on the configuration space, for a total of $2(n+1)$ vector fields. (We have already met v_n and f_n.)

- f_k



This is not a legitimate way of driving the train: we are treating the previous trailers as if they were a hood ornament, with the wheels removed and all angles welded.

- v_k



- One has this basic formula:

$$f_k = v_{k-1} \sin \varphi_k + f_{k-1} \cos \varphi_k$$

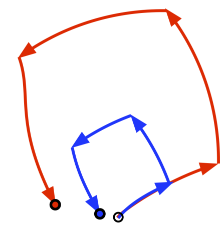
- By repeated use of this formula, one can express any vector, e.g.. f_n, as a linear combination of f_0 and $v_0, v_1, \ldots, v_n$.

Lie brackets

- In Lecture 1, we looked at the problem of parallel parking: maneuvering a vehicle sideways into a location. This led us to the notion of a commutator of two motions.
- We also encountered the notion of Lie bracket of vector fields.
- The following standard formula relates the two notions:

$$[X, Y] = \frac{1}{2} \frac{\partial^2}{\partial t^2}\Big|_0 \left(\mathrm{Fl}^Y_{-t} \circ \mathrm{Fl}^X_{-t} \circ \mathrm{Fl}^Y_t \circ \mathrm{Fl}^X_t \right).$$

- $[X, Y] = \frac{1}{2} \frac{\partial^2}{\partial t^2}\Big|_0 \left(\mathrm{Fl}^Y_{-t} \circ \mathrm{Fl}^X_{-t} \circ \mathrm{Fl}^Y_t \circ \mathrm{Fl}^X_t \right)$
- Here X and Y are two vector fields. The motion Fl^X_t is the *flow* along X: starting at each point of X, integrate the vector field from time 0 to time t; this determines where the point should be moved. The expression in parentheses is a commutator.



- The formula says that Lie bracket is akin to an "infinitesimal commutator." Thus Lie brackets are an important idea in control theory.

- One natural question to ask is: if we are given a distribution $\mathcal{D}$ on a manifold (a subbundle of the tangent bundle) what other vector fields can we generate from the ones in the distribution?
- We think of the vector fields within the distribution as the ones we can realize directly by single motions, and the ones we obtain via Lie bracketing are those which we can obtain by combining motions.

- Recall the *Lie squares sequence*

$$\mathcal{D} = \mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{D}_3 \subset \cdots$$

in which each bundle is the Lie square of the previous bundle:

$$\mathcal{D}_i = (\mathcal{D}_{i-1})^2 = \mathcal{D}_{i-1} + [\mathcal{D}_{i-1}, \mathcal{D}_{i-1}].$$

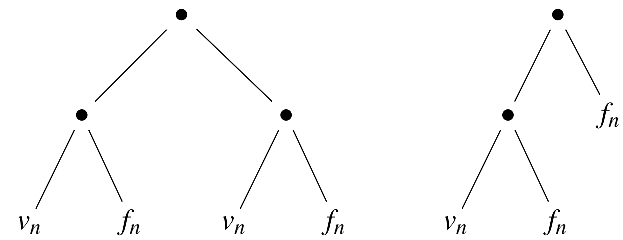
- Recall that a *Goursat distribution* is one for which the rank increases by one at each step until we reach the tangent bundle.
- This is exactly the situation on the configuration space for a truck with n trailers. At each point of the configuration space we have two vectors v_n and f_n, the infinitesimal motions of the train caused by turning and driving the truck, respectively. They fit together into a rank 2 distribution, and it is Goursat.

- Why is it a Goursat distribution?
- By bracketing v_n and f_n, we obtain a third independent vector field. In the distribution spanned by the three vector fields we find both v_{n-1} and f_{n-1}. Thus we now have control of the first trailer, and as if it were the truck.
- Repeat this argument for n steps.

- Note that when we're working in $\mathcal{D}_3$, we're allowed to input four vector fields into calculations, e.g.,

$$\Big[[v_n, f_n], [v_n, f_n]\Big]$$

is a vector field in $\mathcal{D}_3$.
- We can represent this calculation by the tree shown on the left.

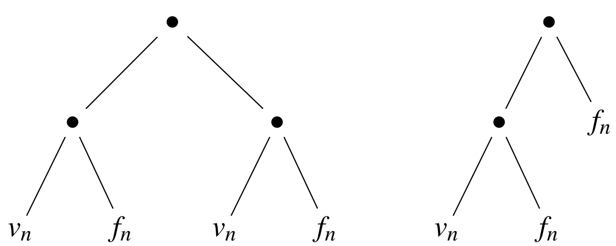


- A more restrictive sort of calculation is represented by the tree on the right, and this leads us to the idea of the *slow growth sequence*

$$\mathcal{D} = \mathcal{D}^1 \subset \mathcal{D}^2 \subset \mathcal{D}^3 \subset \cdots$$

in which

$$\mathcal{D}^i = \mathcal{D}_{i-1} + [\mathcal{D}_{i-1}, \mathcal{D}].$$



- We don't want to assume that these are subbundles; that would be too restrictive. They are *subsheaves* of the sheaf of the tangent bundle. In fact the idea is to study the distribution by studying the associated sequence of ranks, which is a function on the configuration space.
- Generic behavior: for all points representing configurations without any right angles, $\mathcal{D}^i = \mathcal{D}_i$, and thus the *slow growth vector* is simply $(2, 3, 4, \ldots, n+3)$.
- For configurations in which all the bending angles are right angles, the slow growth vector is

$$(2, 3, 4, 4, 5, 5, 5, 6, 6, 6, 6, 6, 7, 7, 7, 7, 7, 7, 7, 7, 8, \ldots),$$

where the number of repetitions is a Fibonacci number.
- Jean's *beta vector* here is $(1, 2, 3, 5, 8, \ldots)$. Its entries tell us when we first reach rank 2, then rank 3, etc.

Singular configurations

- A configuration is *singular* if the slow growth vector is something other than the generic $(2, 3, 4, \ldots, n+3)$.
- Jean provides an elegant recursion for computing the slow growth vector at all points.

- On the second monster we have a chart with coordinates (x, y, y', y''), whereas for the configuration space of the truck with one trailer we use coordinates $(x, y, \varphi_0, \varphi_1)$. For simplicity let's assume $\varphi_1 \geq 0$ and $\varphi_2 \geq 0$.
- Here are the relations between these coordinate systems:

$$\tan\varphi_0 = y'$$

$$\tan\varphi_1 = \frac{y''}{(1+(y')^2)^{3/2}}$$

- Assuming that $y' = 0$, here's the relation for third-order data:

$$\tan\varphi_2 = \frac{y^{(3)} + y'' + (y'')^3}{(1+(y'')^2)^{3/2}}$$

- If we use the chart in which

$$x' = \frac{dx}{dy'} \quad \text{and} \quad x'' = \frac{dx'}{dy'}$$

then here are the relations:

$$\cot\varphi_1 = x'(1+(y')^2)^{3/2}$$

$$\tan\varphi_2 = \frac{-x'' + 1 + (x')^2}{((x')^2+1)^{3/2}} \qquad \text{(assuming } y' = 0\text{)}$$

- Looking at

$$\cot\varphi_1 = x'(1+(y')^2)^{3/2}$$

$$\tan\varphi_2 = \frac{-x'' + 1 + (x')^2}{((x')^2+1)^{3/2}} \qquad \text{(assuming } y' = 0\text{)}$$

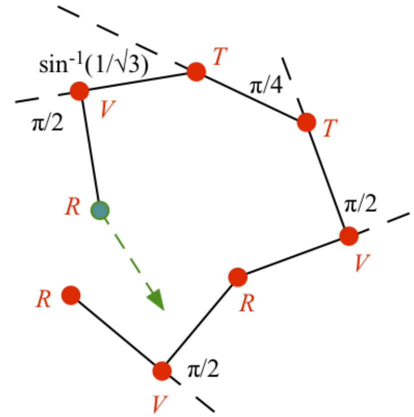
- The divisor at infinity I_2 is defined by the vanishing of x', and thus by $\varphi_1 = \pi/2$.
- Its baby-monster prolongation $I_2[1]$ is defined by the vanishing of both x' and x'', and thus by the additional condition $\varphi_2 = \pi/4$.

- Given a configuration of the train, here's how to read its *RVT* code word from its bending angles:
  - If $\varphi_i = \pm\pi/2$, then write *V* in position *i*. If you see the sequence of special bending angles

$$\pm\left(\pi/2, \quad \pi/4, \quad \sin^{-1}(1/\sqrt{3}), \quad \sin^{-1}(1/\sqrt{4}), \quad \cdots\right)$$

    starting at position *i*, then write *V* in position *i* followed by a sequence of *T*'s for the other special angles.
  - In all other cases write *R*.

- A train with code word *RVRVTTVR*



- The code word *W* of a configuration determines its slow growth vector
  - equivalently, its beta vector $\beta(W) = (\beta_1(W), \beta_2(W), \ldots, \beta_k(W))$.
- Jean's recursion:

$$\begin{aligned} \beta_j(WR) &= \beta_{j-1}(W) \\ \beta_j(WXV) &= \beta_{j-2}(W) + \beta_{j-1}(WX) \\ \beta_j(WXT) &= 2\beta_{j-1}(WX) - \beta_{j-2}(W) \end{aligned}$$

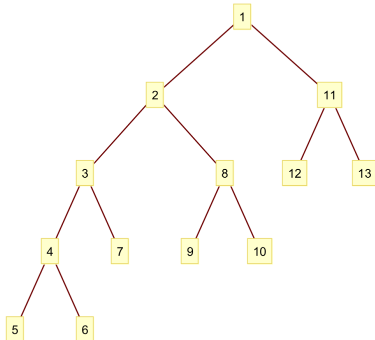
Jean's beta vector

R	1	2					
R	1	2	3				
R	1	2	3	4			
R	1	2	3	4	5		
R	1	2	3	4	5	6	
R	1	2	3	4	5	6	7
R	1	2					
R	1	2	3				
V	1	2	3	5			
V	1	2	3	5	8		
V	1	2	3	5	8	13	
V	1	2	3	5	8	13	21
R	1	2					
R	1	2	3				
V	1	2	3	5			
T	1	2	3	4	7		
T	1	2	3	4	5	9	
V	1	2	3	5	7	9	16

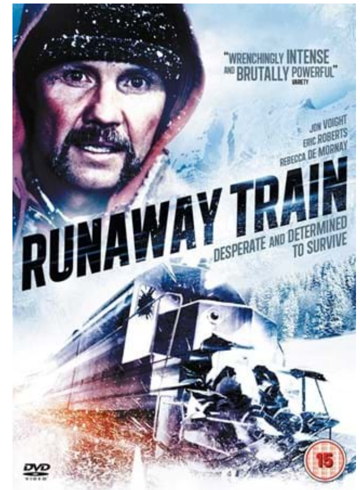
Slow growth vector

	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*	*21*
R	2	3																			
R	2	3	4																		
R	2	3	4	5																	
R	2	3	4	5	6																
R	2	3	4	5	6	7															
R	2	3	4	5	6	7	8														
R	2	3																			
R	2	3	4																		
V	2	3	4	4	5																
V	2	3	4	4	5	5	5	6													
V	2	3	4	4	5	5	5	6	6	6	6	6	7								
V	2	3	4	4	5	5	5	6	6	6	6	6	7	7	7	7	7	7	7	7	8
R	2	3																			
R	2	3	4																		
V	2	3	4	4	5																
T	2	3	4	5	5	5	6														
T	2	3	4	5	6	6	6	6	7												
V	2	3	4	4	5	5	6	6	7	7	7	7	7	7	7	8					

- Together with Corey Shanbrom, we're in the midst of a project which we think will explain Jean's recursion by developing a theory of calculational trees.
- For example, we speculate that this tree should be associated to the code word *RRVT*. Note that the number of leaves is 7, which is the last entry in Jean's beta vector.



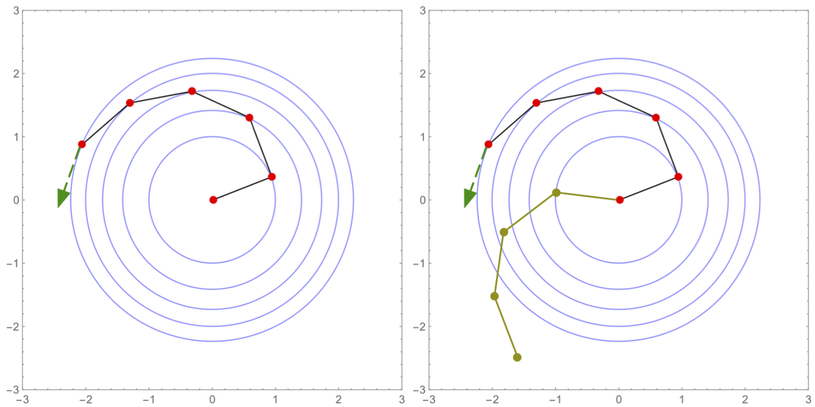
The runaway train



- Alejandro Bravo-Doddoli has reported to you about his paper with García-Naranjo. We'll just point out how the special strata on the monster come up in their construction.
- Suppose:
  - Each trailer (including the truck) has an equal point mass.
  - The train is in a certain configuration.
  - The truck is set into motion with a certain initial velocity and initial angular velocity.
  - Thereafter the truck and trailers move according to Newtonian mechanics.
- We want to know how it moves. This is an initial value problem (not a control theory problem).

- In Newtonian mechanics, the energy is conserved, so in our analysis it makes sense to specify the level of energy and then to examine the possible trajectories at that level.
- Bravo-Doddoli and García-Naranjo show that, on a trajectory, the angular velocity of the truck is constant, so let's also fix that constant and assume it's not zero.

- We already know many periodic solutions:



- The rigid train leads to a single periodic trajectory, while the embellishments lead to entire tori of periodic trajectories.
- Thus there are $n+1$ *critical values* of the energy at which we find this abundance of periodic solutions.
- Bravo-Doddoli and García-Naranjo do a bifurcation analysis, showing that at each of these critical values, there is a change in the dynamics of the system.