Aspects of the Monster Tower Construction:
Geometric, Combinatorial, Mechanical, Enumerative

Lecture 2: Combinatorial Aspects

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Charts and coordinates

- Begin with an open set U on M with coordinates $x_1, \ldots, x_m$ so that at each point the differentials $dx_1, \ldots, dx_m$ form a basis of the cotangent space.
- The monster tower over U is a tower of bundles with fiber $\mathbb{P}^{m-1}$. We explain here a systematic way of naming charts and coordinates for the spaces in the tower. We'll assume $m = 3$.

- At a nonsingular point of a curve, its curvilinear data are the values of

$$x_1 \qquad x_2 \qquad x_3$$

$$x_2' := \frac{dx_2}{dx_1} \qquad x_3' := \frac{dx_3}{dx_1}$$

$$x_2'' := \frac{dx_2'}{dx_1} \qquad x_3'' := \frac{dx_3'}{dx_1}$$

$$x_2^{(3)} := \frac{dx_2''}{dx_1} \qquad x_3^{(3)} := \frac{dx_3''}{dx_1}$$

etc.

- To record the data of singular curves, we need to
  - turn some of these quantities upside-down, and
  - work with derivatives with respect to quantities other than the three coordinates on the base.
- There will be 3^k charts (where k is the level), and each one is determined by a sequence of choices. We'll show one possibility.

- Coordinates for chart $\mathcal{C}(3)$ on $M(1)$ — using 5 unique coordinates

$$x_1 \qquad\qquad x_2 \qquad\qquad x_3$$

$$x_1(3) := \frac{dx_1}{dx_3} \qquad x_2(3) := \frac{dx_2}{dx_3} \qquad \boxed{x_3(3) := x_3}$$

- The box indicates that we have introduced a (conveniently) redundant name. The three coordinates in the bottom row are called *active coordinates*; to continue, we choose one of them, and differentiate the two other active coordinates with respect to it.

- Coordinates for chart $\mathcal{C}(32123)$ on $M(5)$ — using 13 unique coordinates

$$x_1 \qquad\qquad\qquad x_2 \qquad\qquad\qquad x_3$$

$$x_1(3) := \frac{dx_1}{dx_3} \qquad x_2(3) := \frac{dx_2}{dx_3} \qquad \boxed{x_3(3)}$$

$$x_1(32) := \frac{dx_1(3)}{dx_2(3)} \qquad \boxed{x_2(32)} \qquad x_3(32) := \frac{dx_3(3)}{dx_2(3)}$$

$$\boxed{x_1(321)} \qquad x_2(321) := \frac{dx_2(32)}{dx_1(32)} \qquad x_3(321) := \frac{dx_3(32)}{dx_1(32)}$$

$$x_1(3212) := \frac{dx_1(321)}{dx_2(321)} \qquad \boxed{x_2(3212)} \qquad x_3(3212) := \frac{dx_3(321)}{dx_2(321)}$$

$$x_1(32123) := \frac{dx_1(3212)}{dx_3(3212)} \qquad x_2(32123) := \frac{dx_2(3212)}{dx_3(3212)} \qquad \boxed{x_3(32123)}$$

- Each function extends to a rational function on the entire monster space, and there are no ambiguities in the names.
- These coordinate systems are essentially due to Lejeune-Jalabert (except for the redundant names).

- Alternative notation when the base is 2-dimensional:

$$x$$

$$y$$

$$y' = \frac{dy}{dx}$$

$$y'' = \frac{dy'}{dx}$$

$$x' = \frac{dx}{dy''}$$

$$x'' = \frac{dx'}{dy''}$$

$$y^{(3)} = \frac{dy''}{dx''}$$

- It's simpler, but it carries less information and is ambiguous.

- At each step our choice can be framed as follows:
  - Differentiate again with respect to the same coordinate as in the previous step (make the **Regular choice**).
  - Differentiate with respect to the coordinate which has just been introduced (make the **Critical choice**).
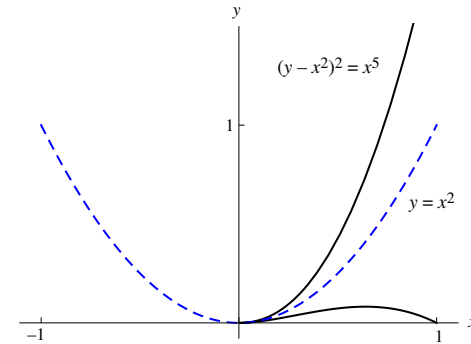
$$y' = \frac{dy}{dx} \qquad R$$

$$y'' = \frac{dy'}{dx} \qquad R$$

$$x' = \frac{dx}{dy''} \qquad C$$

$$x'' = \frac{dx'}{dy''} \qquad R$$

$$y^{(3)} = \frac{dy''}{dx''} \qquad C$$

Lifting (prolonging)

- Robert Bryant: "Intuitively, prolongation is just differentiating the equations you have and then adjoining those equations as new equations in the system."

- Consider a nonsingular point p of a curve C in M. We can associate to p the point of $\mathbb{P}TM$ representing the tangent direction of C at p. Thus, away from singularities, we have a curve $C(1)$ in $\mathbb{P}TM$, the *prolongation* or *lift*.
- In turn, we can lift this curve to the next level (arguing that in fact all of its tangent directions are focal), and continue upward through the tower, obtaining lifts $C(2)$, $C(3)$, etc.

- We also want to lift a singular curve C in M, and we do so by fiat: we lift at all nonsingular points of C, and then take the closure.
  - This may give us several points over a singular point of C.
  - Even if it gives us a single point, the nature of the singularity will change, and we may even get a nonsingular point on the lift.

- Example with a parametrized curve: the *ramphoid cusp*

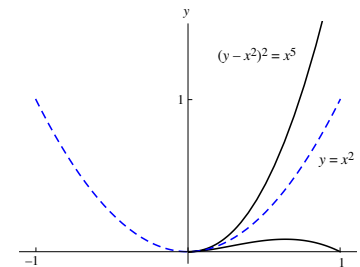$$\begin{cases} x = t^2 \\ y = t^4 + t^5 \end{cases}$$

- Calculating the lift:

$$x = t^2$$
$$y = t^4 + t^5$$
$$y' = \frac{dy}{dx} = \frac{4t^3 + 5t^4}{2t} = 2t^2 + \frac{5}{2}t^3$$
$$y'' = \frac{dy'}{dx} = \frac{4t + \frac{15}{2}t^2}{2t} = 2 + \frac{15}{4}t$$
$$x' = \frac{dx}{dy''} = \frac{8}{15}t.$$

- These equations parametrize a curve in the third monster.

- Setting $t = 0$, we find one point over the origin: $(x, y, y', y'', x') = (0, 0, 0, 2, 0)$.

- The (generalized) tangent line is horizontal.
- We have the same data up to second order as the parabola $y = x^2$.
- Its third-order datum is infinite.

- Example: the *cuspidal cubic* $y^2 = x^3$
- Implicit differentiation yields

$$2yy' = 3x^2.$$

- Note that these two equations don't give an irreducible curve, since over the origin they allow any value for y'.

- To remove this spurious component, we remark that, assuming we are away from the origin, we have

$$x = \frac{4}{9}(y')^2 \quad \text{and} \quad y = \frac{8}{27}(y')^3.$$

Thus these equations should also be used in defining the lift.

- To continue the calculation for one more step, we use $x' := dx/dy'$, and calculate that

$$x' = \frac{8}{9}y'.$$

- Thus the second-order datum at the origin is infinite. If, loosely speaking, we think of this as curvature, we're claiming that as we approach the origin on this curve, its curvature goes to infinity.
- In the figure, note how the osculating circles shrink to a point as we approach the origin.
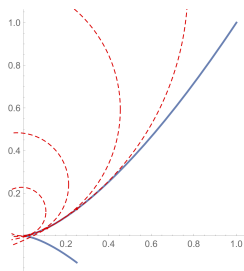
- Going in the opposite direction: given a point on the kth monster, we want to generate a curve for which it is the curvilinear data.
- Montgomery and Zhitomirskii call this process *Legendrization*.

- Example: in the chart with coordinates

$$y' = \frac{dy}{dx} \qquad x' = \frac{dx}{dy'} \qquad x'' = \frac{dx'}{dy'} \qquad y'' = \frac{dy'}{dx''}$$

let's find a curve whose lift goes through the point
$(x, y, y', x', x'', y'') = (0, 0, 0, 0, 1, 1)$.

$$y'' = 1 + t$$

$$x'' = 1 + t$$

$$y' = \int y'' dx'' = \int (1 + t)\, dt = \tfrac{1}{2}(1+t)^2 - \tfrac{1}{2}$$

$$x' = \int x'' dy' = \tfrac{1}{3}(1+t)^3 - \tfrac{1}{3}$$

$$x = \int x' dy' = \tfrac{1}{15}(1+t)^5 - \tfrac{1}{6}(1+t)^2 + \tfrac{1}{10}$$

$$y = \int y' dx = \tfrac{1}{42}(1+t)^7 - \tfrac{1}{30}(1+t)^5 - \tfrac{1}{24}(1+t)^4 + \tfrac{1}{12}(1+t)^2 - \tfrac{9}{280}$$

Coarse stratification

- We now apply the basic construction in a different way. Again suppose that $\mathcal{B}$ is a rank b subbundle of the tangent bundle TM.
- Then the relative tangent bundle $\mathcal{V} = T(\widetilde{M}/M)$ is a subbundle of $\widetilde{\mathcal{B}}$; its fiber consists of vectors mapping to zero, sometimes called *vertical vectors*. Its rank is $b - 1$.
- Applying the basic construction to $\mathcal{V}$, we obtain the space $\mathbb{P}\mathcal{V}$ carrying its focal bundle $\widetilde{\mathcal{V}}$, again of rank $b - 1$.
- Since $\mathcal{V}$ is a subbundle of $\widetilde{\mathcal{B}}$, the space $\mathbb{P}\mathcal{V}$ is naturally a subset of $\mathbb{P}\widetilde{\mathcal{B}}$. In fact, each fiber of $\mathbb{P}\mathcal{V}$ is a hyperplane inside the $\mathbb{P}^{b-1}$-fiber of $\mathbb{P}\widetilde{\mathcal{B}}$. Thus $\mathbb{P}\mathcal{V}$ is a divisor (codimension one subvariety) of $\mathbb{P}\widetilde{\mathcal{B}}$. We call it a *baby monster*.

- Here's where we want to apply the baby monster construction: to the monster spaces over some base manifold or variety M.
- We can do this to any projection map in the monster tower

$$M(j-1) \xrightarrow{\pi_{j-1}} M(j-2),$$

and what we obtain is a divisor on $M(j)$, which we call the jth *divisor at infinity* and denote by I_j.
- If the dimension of the base is m, then by construction I_j carries a bundle of rank $m - 1$. We can iterate this construction, building a tower of $\mathbb{P}^{m-2}$-bundles over $M(j-1)$.

- Here is our notation:

$$\cdots \to I_j[3] \to I_j[2] \to I_j[1] \to I_j[0] = I_j \quad \to M(j-1)$$

- Note that:
  - I_j is a divisor on $M(j)$
  - $I_j[1]$ has codimension 2 in $M(j+1)$
  - $I_j[2]$ has codimension 3 in $M(j+2)$
  - etc.

  (If $m = 2$, then all these maps are isomorphisms.)

Charts and coordinates
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Lifting (prolonging)
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Coarse stratification
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Code words
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Moduli
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- What's the geometric meaning of these loci?
- Using either our interpretation of the coordinate systems, or reflecting upon how the lift of a curve could possibly have a vertical tangent, we realize that the divisor at infinity consists of those curvilinear data points for which we consider the jth-order data to be infinite.

Charts and coordinates
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Lifting (prolonging)
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Coarse stratification
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Code words
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Moduli
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- For example, we have seen that the lift of the cuspidal cubic curve $y^2 = x^3$ is tangent to the fiber of the first monster over the origin. Thus the second lift meets the divisor at infinity.

Charts and coordinates
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Lifting (prolonging)
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Coarse stratification
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Code words
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Moduli
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- Repeating the example of the ramphoid cusp, note the second lift is tangent to the fiber of $M(2)$ over $M(1)$:

$$x = t^2$$
$$y = t^4 + t^5$$
$$y' = 2t^2 + \frac{5}{2}t^3$$
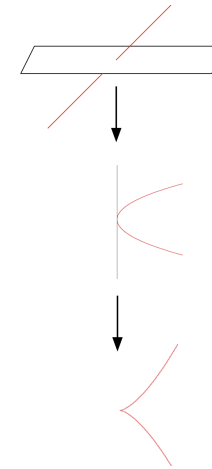$$y'' = 2 + \frac{15}{4}t$$

- We can see that the third lift hits I_3:

$$y''' = \frac{dy''}{dx} = \frac{15}{8t} \quad \text{(goes to infinity)}$$

Alternatively,

$$x' = \frac{dx}{dy''} = \frac{8}{15}t \quad \text{(goes to zero)}$$

Charts and coordinates
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Lifting (prolonging)
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Coarse stratification
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Code words
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Moduli
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- As this example suggests, it's straightforward to use coordinates to tell when you're on a divisor at infinity. We work this out in full detail in a recent paper with Castro and Shanbrom. In this example, the divisor at infinity is the locus $x' = 0$.

- What about $I_j[1]$? What's its geometric meaning?
- Suppose that when you lift your curve to $M(j-1)$, you are not just tangent to a fiber over $M(j-2)$, but tangent to second order. Then when you lift to $M(j)$, you won't just hit I_j; you'll be tangent to it. And when you lift yet again, then you'll be on $I_j[1]$.

- To compare all these divisors at infinity and their associated baby monsters, let's pull everything back to our highest level k. The complete inverse image of I_j will be denoted in the same way, but now it's a divisor up on $M(k)$.
- With this convention, we now have a nest

$$I_j \supset I_j[1] \supset I_j[2] \supset \cdots \supset I_j[k-j].$$

- We can also form intersections of spaces taken from different nests: we consider the *intersection locus*

$$I_W := \bigcap_{j=2}^{k} I_j[n_j - 1]$$

for some specified nonnegative integers n_j (with the convention that $I_j[-1]$ is all of $M(k)$).

- The main theorem of our paper with Castro and Shanbrom says this:
  - This is a transverse intersection:

$$I_W := \bigcap_{j=2}^{k} I_j[n_j - 1]$$

  - Its codimension (if it's nonempty) is $n_2 + \cdots + n_k$.
- It also explains when the intersection is nonempty, and that naturally leads to our next topic.

Code words

- The letter W in

$$I_W := \bigcap_{j=2}^{k} I_j[n_j - 1]$$

represents the *code word* of the intersection locus. It records, in a different way, which spaces we are intersecting.
- The alphabet for the code consists of all symbols V_A, where A is a finite subset of the integers strictly greater than 1. Although this is an infinite alphabet, the rules for creating a valid word will imply that there are only finitely many words of each specified length.
- To be consistent with prior usage, we will use R in place of $V_\emptyset$.

- The rules for creating a code word:
  1. The first symbol must be R.
  2. Immediately following the symbol V_A, one may put any symbol V_B, where either B is a subset of A, or B is a subset of $A \cup \{j\}$, with j being the position of the symbol.
  3. The cardinality of A is less than m.
- Note that j cannot appear in a subscript prior to position j.
- Example at level 8:

$$W = RV_2 V_{23} V_{23} V_{25} V_5 V_5 V_5.$$

- How the code word specifies an intersection locus (via an example):

$$W = RV_2 V_{23} V_{23} V_{25} V_5 V_5 V_5.$$

- The number 2 first appears in position 2, and then appears in three subsequent positions. This means that one of the spaces used in the intersection is $I_2[3]$. Similarly, we use $I_3[1]$.
- The number 4 never appears in a subscript. Thus we can omit I_4, i.e., write $I_4[-1]$; similarly we can omit I_6, I_7, and I_8.
- Here is the result:

$$I_W = I_2[3] \cap I_3[1] \cap I_4[-1] \cap I_5[3] \cap I_6[-1] \cap I_7[-1] \cap I_8[-1]$$
$$= I_2[3] \cap I_3[1] \cap I_5[3].$$

- Reiterating: If $W = RV_2 V_{23} V_{23} V_{25} V_5 V_5 V_5$, then

$$I_W = I_2[3] \cap I_3[1] \cap I_4[-1] \cap I_5[3] \cap I_6[-1] \cap I_7[-1] \cap I_8[-1]$$
$$= I_2[3] \cap I_3[1] \cap I_5[3].$$

- Is this intersection locus nonempty? The answer depends on the dimension of the base. Note that the third rule for code words limits the length of the subscripts. Thus this intersection locus is nonempty if the base has dimension at least 3; over a surface it's empty.
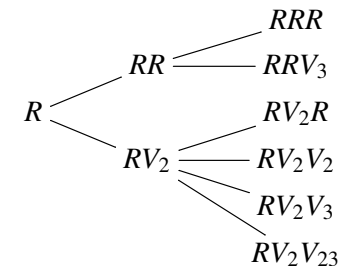
- Code words up to length 3 (if the dimension of the base is at least 3):

$$
\begin{array}{ccc}
 & & RRR \\
 & RR & RRV_3 \\
R & & RV_2 R \\
 & RV_2 & RV_2 V_2 \\
 & & RV_2 V_3 \\
 & & RV_2 V_{23}
\end{array}
$$

- Over a surface, the last code word should be omitted; thus there are five intersection loci.

- When the base is a surface, note that at each position j of the code word there are at most three possibilities:
  1. Use R.
  2. Use V_j.
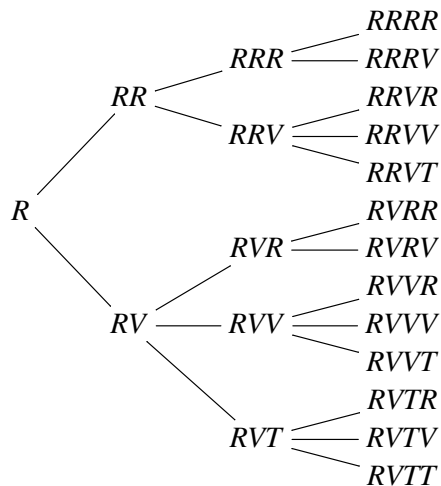 3. Repeat the previous symbol (assumed not to be R).

  This leads to the alternative RVT code, originally developed by the differential geometers.
- The five intersection loci at level three, with representative curve germs:

$$
\begin{array}{c|c|c|c|c}
RRR & RRV & RVR & RVV & RVT \\
RRR & RRV_3 & RV_2R & RV_2V_3 & RV_2V_2 \\
y = 0 & y^2 = x^5 & y^2 = x^3 & y^3 = x^5 & y^3 = x^4
\end{array}
$$

- How many intersection loci are there? To answer this, one can work out a simple recursion.
- In the case of a surface, the recursion immediately implies that the number of code words of specified length is a Fibonacci number: there are 2 intersection loci at level two, 5 at level three, 13 at level four, etc.

- RVT code words up to length 4

```
                                      RRRR
                          RRR  ——————  RRRV
              RR                        RRVR
                          RRV  ——————  RRVV
                                        RRVT
                                        RVRR
                          RVR  ——————  RVRV
R                                       RVVR
              RV  ——————  RVV  ——————  RVVV
                                        RVVT
                                        RVTR
                          RVT  ——————  RVTV
                                        RVTT
```

Moduli

- Thus far we have only discussed discrete or coarse aspects of the monster spaces. A few researchers, most notably Mormul, have devoted considerable attention to understanding their moduli.
- We're thinking about equivalence classes of plane curve germs or (for Mormul) equivalence classes of Goursat germs, where (in the case of curves) we allow arbitrary diffeomorphisms at a point of the base manifold M.
- If two curve germs give different code words, then certainly they are inequivalent. But what if they have the same code word?

- We find it surprising that in fact there are moduli, i.e., there are continuous families of inequivalent curve germs (or Goursat germs) with the same code word.

- This is wide-open territory. Currently we have no general unifying insight as to where and why moduli should appear.