THE ZIEGLER SPECTRUM OF A LOCALLY COHERENT GROTHENDIECK CATEGORY

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In memory of Maurice Auslander

Let R be a ring (associative, with identity) and denote by mod-R the category of finitely presented right (unital) modules over R. The prototype of a locally coherent Grothendieck category is the category

$$_R\mathscr{C} := (\text{mod-}R, \text{Ab})$$

whose objects are the additive functors F: mod-$R \to$ Ab (Ab denotes the category of abelian groups) and whose morphisms are the natural transformations. The methods of this paper will illustrate how the category $_R\mathscr{C}$ is used to study *left* R-modules. Indeed, we refer to the category $_R\mathscr{C}$ as the category of generalized left R-modules on account of the right exact, fully faithful functor $_RM \mapsto - \otimes_R M$ from the category R-Mod of all left R-modules to $_R\mathscr{C}$.

Let fp-($_R\mathscr{C}$) denote the full subcategory of the finitely presented objects of $_R\mathscr{C}$. For example (cf. [**1**, Lemma 6.1]), if $_RM$ is a finitely presented left R-module, then $- \otimes_R M$ is an object of fp-($_R\mathscr{C}$). It is a key observation of Auslander [**1**, Theorem 2.2] that the category fp-($_R\mathscr{C}$) is abelian. Equivalently (cf. Theorem 1.6), every finitely presented object $B \in {}_R\mathscr{C}$ is *coherent,* that is, B is finitely presented and every finitely generated subobject of B is also finitely presented. It follows from Yoneda's Lemma that $_R\mathscr{C}$ is locally finitely presented and therefore locally coherent (see definition below).

Ziegler [**37**] associates to the ring R a topological space whose points are the isomorphism types of the pure-injective indecomposable left R-modules $_RU$. This space is homeomorphic, via the function $_RU \to - \otimes_R U$, to the topological space Zg($_R\mathscr{C}$) whose points are the isomorphism types of the injective indecomposable objects of the category $_R\mathscr{C}$ and an open basis of Zg($_R\mathscr{C}$) is given by the collection of subsets

$$\mathcal{O}(C) := \{E \in \text{Zg}(_R\mathscr{C}) \colon \ \text{Hom}_{_R\mathscr{C}}(C, E) \neq 0\}$$

as C ranges over the coherent objects of $_R\mathscr{C}$. This topological space is called the *Ziegler spectrum* of $_R\mathscr{C}$.

The notion of a coherent object makes sense in any Grothendieck category $\mathscr{C}$ and such a category is said to be *locally coherent* if every object $X \in \mathscr{C}$ may be represented as a direct limit

$$X \cong \varinjlim C_i$$

of coherent objects C_i of $\mathscr{C}$. Locally coherent categories were introduced by Roos [**31**]. The full subcategory of coherent objects of $\mathscr{C}$ is denoted by coh-$\mathscr{C}$. The

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Ziegler spectrum $\mathrm{Zg}(\mathscr{C})$ of a locally coherent Grothendieck category $\mathscr{C}$ is defined as for ${}_R\mathscr{C}$.

Although this paper contains no model theory *per se,* the point is to survey in the category-theoretic idiom model-theoretic methods (cf. [**27**, **37**]) used in the study of modules. The categorical setting for these methods is that of a locally coherent Grothendieck category. As part of this paper's introduction, we shall recall a portion of Gabriel's theory [**12**] of localization for this special case. In the latter sections, we apply the methods developed in the paper to treat recent results of Crawley-Boevey [**8**].

The main result of this paper is a Nullstellensatz for locally coherent Grothendieck categories. To understand its statement, recall that a full subcategory $\mathscr{S} \subseteq \text{coh-}\mathscr{C}$ is called a Serre subcategory if for every short exact sequence

$$0 \to A \to B \to C \to 0$$

in coh-$\mathscr{C}$, we have that $B \in \mathscr{S}$ if and only if $A, C \in \mathscr{S}$. Serre subcategories of coh-$\mathscr{C}$ arise in the following way. An object X of $\mathscr{C}$ is called *coh-injective* if $\mathrm{Ext}^1_{\mathscr{C}}(C, X) = 0$ for every coherent object C. (In this setting, coh-injectivity is equivalent to the more familiar notion of fp-injectivity [**19**].) Then the subcategory of coh-$\mathscr{C}$,

$$\mathscr{S}(X) := \{C \in \text{coh-}\mathscr{C}\colon\ \mathrm{Hom}_{\mathscr{C}}(C, X) = 0\},$$

is a Serre subcategory. In ${}_R\mathscr{C}$, the coh-injective objects have been characterized [**19**, Theorem B.15] as precisely the objects of the form $- \otimes_R M$ where ${}_RM$ is a left R-module. In this way R-Mod is recovered within ${}_R\mathscr{C}$ and a Serre subcategory of coh-$\mathscr{C}$ is associated to every left R-module.

THEOREM 3.8. *Let $\mathscr{C}$ be a locally coherent Grothendieck category. There is an inclusion-preserving bijective correspondence between the Serre subcategories $\mathscr{S}$ of* coh-$\mathscr{C}$ *and the open subsets $\mathscr{O}$ of* $\mathrm{Zg}(\mathscr{C})$. *This correspondence is given by the functions*

$$\mathscr{S} \mapsto \mathscr{O}(\mathscr{S}) := \bigcup_{C \in \mathscr{S}} \mathscr{O}(C)$$

and

$$\mathscr{O} \mapsto \mathscr{S}_{\mathscr{O}} := \{C \in \text{coh-}\mathscr{C}\colon\ \mathscr{O}(C) \subseteq \mathscr{O}\}$$

which are mutual inverses.

The advantage of working in the context of locally coherent Grothendieck categories is that one can show that every locally closed subset $I \cap \mathscr{O}$ (here I denotes a closed set) of the Ziegler spectrum $\mathrm{Zg}(\mathscr{C})$ of a locally coherent Grothendieck category $\mathscr{C}$ is homeomorphic to the Ziegler spectrum of a subcategory of $\mathscr{C}$ which is also locally coherent Grothendieck. For the case of an open set $\mathscr{O}$ or a closed set I, this is done as follows.

$\mathscr{O}$. By Theorem 3.8, there is a Serre subcategory $\mathscr{S} \subseteq \text{coh-}\mathscr{C}$ such that $\mathscr{O} = \mathscr{O}(\mathscr{S})$. Let $\vec{\mathscr{S}}$ [**9**] denote the full subcategory of $\mathscr{C}$ of those objects Y which may be represented as a direct limit

$$Y \cong \varinjlim S_i$$

of objects from $\mathcal{S}$. Then $\vec{\mathcal{S}}$ is a locally coherent category whose Ziegler spectrum $\mathrm{Zg}(\vec{\mathcal{S}})$ is homeomorphic to $\mathcal{O}(\mathcal{S})$.

I. If $I \subseteq \mathrm{Zg}(\mathcal{C})$ is closed, let $\mathcal{S} \subseteq \text{coh-}\mathcal{C}$ be such that $\mathcal{O}(\mathcal{S}) = \mathrm{Zg}(\mathcal{C}) \setminus I$. To the subcategory $\vec{\mathcal{S}} \subseteq \mathcal{C}$ is associated the torsion functor

$$t_{\mathcal{S}}\colon\ \mathcal{C} \to \vec{\mathcal{S}}$$

which assigns to an object X of $\mathcal{C}$ its maximal subobject from $\vec{\mathcal{S}}$. The functor $t_{\mathcal{S}}$ is left exact and the object X is called $\mathcal{S}$*-closed* if $t_{\mathcal{S}}(X) = t^1_{s\mathcal{S}}(X) = 0$ where $t^1_{\mathcal{S}}$ denotes first higher derived functor of $t_{\mathcal{S}}$. The full subcategory $\mathcal{C}/\vec{\mathcal{S}}$ of $\mathcal{S}$-closed objects is a locally coherent Grothendieck category whose Ziegler spectrum $\mathrm{Zg}(\mathcal{C}/\vec{\mathcal{S}})$ is homeomorphic to I.

Thus every Serre subcateogry $\mathcal{S}$ of coh-$\mathcal{C}$ gives a partition of the Ziegler spectrum of $\mathcal{C}$,

$$\mathrm{Zg}(\mathcal{C}) \cong \mathrm{Zg}(\vec{\mathcal{S}}) \,\dot\cup\, \mathrm{Zg}(\mathcal{C}/\vec{\mathcal{S}}),$$

into an open and a closed set.

We shall consider three general sets of points of the left Ziegler spectrum $\mathrm{Zg}({}_R\mathcal{C})$ of the ring R.

(1) The injective indecomposables, that is, the points $-\otimes_R E$ where ${}_RE$ is an indecomposable injective R-module.

(2) If ${}_RM$ is a finitely presented R-module with local endomorphism ring, then the injective envelope $E(-\otimes_R M)$ of $-\otimes_R M$ is a point of $\mathrm{Zg}({}_R\mathcal{C})$. If the ring R contains a complete local noetherian ring in its centre and R is finitely generated as a module over this subring, then $-\otimes_R M = E(-\otimes_R M)$ is already an injective object of ${}_R\mathcal{C}$. The set of such points is dense in the left Ziegler spectrum of such an R.

Similar considerations apply to an artin algebra Λ, so that if ${}_\Lambda M$ is a finitely generated indecomposable Λ-module, then $-\otimes_\Lambda M$ is a point of $\mathrm{Zg}({}_\Lambda\mathcal{C})$. By [**27**, Corollary 13.4], these are precisely the isolated points of $\mathrm{Zg}({}_\Lambda\mathcal{C})$.

(3) An *endofinite* R-module ${}_RM$ is one that has finite length as a module over its endomorphism ring $\mathrm{End}_R\, M$. This finite length is called the *endolength* of ${}_RM$. Every endofinite indecomposable R-module is a point of the left Ziegler spectrum of R. We prove (cf. Corollary 9.4) that for every natural number n, the set $\mathrm{Zg}_n({}_R\mathcal{C})$ of points of endolength at most n is closed. In Theorem 9.5, the closed subset $\mathrm{Zg}_1({}_R\mathcal{C})$ is shown to be the field spectrum of R in the sense of Cohn [**7**] endowed with the constructible topology. The work [**8**] of Crawley-Boevey shows that for an infinite artin algebra Λ, the Second Brauer–Thrall Conjecture is equivalent to the existence of a non-isolated endofinite point in $\mathrm{Zg}({}_\Lambda\mathcal{C})$.

The Nullstellensatz (Theorem 3.8) assumes the rôle of Ziegler's [**37**, Lemma 4.7] which is, in the model theory of modules, the most often applied form of Gödel's Compactness Theorem. The following is an example of such an application.

Corollary 3.9 (Ziegler [**37**, Theorem 4.9]). *Let $\mathcal{C}$ be a locally coherent Grothendieck category. An open subset $\mathcal{O}$ of* $\mathrm{Zg}(\mathcal{C})$ *is quasi-compact if and only if it is one of the basic open subsets $\mathcal{O}(C)$ where C is a coherent object of $\mathcal{C}$.*

For the ring R, the forgetful functor $-\otimes_R R\colon$ mod-$R \to \mathrm{Ab}$ is a coherent object of ${}_R\mathcal{C}$ and therefore the left Ziegler spectrum of R, $\mathrm{Zg}({}_R\mathcal{C}) = \mathcal{O}(-\otimes_R R)$, is a quasi-compact topological space. Quasi-compactness is the property of the Ziegler spectrum of a ring used in proving the existence of large (not finitely generated) modules. If the ring is an artin algebra Λ, which is not of finite representation type, then an accumulation point of the set of isolated points witnesses the following result of Auslander.

PROPOSITION 7.9 (Auslander [**3**; **27**, Corollary 13.4]). *If Λ is an artin algebra that is not of finite representation type, then there exists a (pure-injective) indecomposable Λ-module which is not finitely generated.*

Similarly, an accumulation point of the isolated points of $\mathrm{Zg}_n({}_\Lambda\mathscr{C})$ witnesses the following result of Crawley-Boevey.

THEOREM 9.6 (Crawley-Boevey [**8**, Theorem 9.6]). *Let Λ be an artin algebra and n a natural number. If there are infinitely many finitely generated indecomposable Λ-modules of endolength at most n, then there is an indecomposable Λ-module of endolength at most n which is not finitely generated.*

The categorical duality $D\colon (\text{coh-}{}_R\mathscr{C})^{\text{op}} \to \text{coh-}\mathscr{C}_R$ of Auslander [**4**] and Gruson and Jensen [**16**] is described in § 5. In the model theory of modules, this corresponds to elementary duality, introduced by Prest [**27**, Chapter 8] and developed in [**17**]. If $\mathscr{S} \subseteq \text{coh-}{}_R\mathscr{C}$ is a Serre subcategory, we let $D\mathscr{S} = \{DS\colon S \in \mathscr{S}\} \subseteq \text{coh-}\mathscr{C}_R$ denote the dual Serre subcategory of $\mathscr{C}_R$.

THEOREM 5.5 [**17**, Proposition 4.4]. *Let R be a ring. There is an inclusion-preserving bijective correspondence between the Serre subcategories of* coh-$({}_R\mathscr{C})$ *and those of* coh-$(\mathscr{C}_R)$ *given by*

$$\mathscr{S} \mapsto D\mathscr{S}.$$

The induced map $\mathcal{O}(\mathscr{S}) \mapsto \mathcal{O}(D\mathscr{S})$ gives an isomorphism between the topologies, that is, the respective algebras of open sets, of the left and right Ziegler spectra of R.

In the final section, Theorem 3.8 is applied to the characters [**8**] of Crawley-Boevey. These generalize the Sylvester rank functions of Schofield [**33**]. The Grothendieck group $K_0(\text{coh-}\mathscr{C})$ is endowed with a pre-order and a *character* $\xi\colon K_0(\text{coh-}\mathscr{C}) \to Z$ (the integers) is defined to be any order-preserving group homomorphism. An *irreducible* character is one that is not the sum of two non-zero characters. We give a proof of the following result of Crawley-Boevey.

THEOREM 8.6 (Crawley-Boevey [**8**, Theorem 5.2]). *Every character*

$$\xi\colon K_0(\text{Coh-}\mathscr{C}) \to Z$$

is expressible uniquely as a sum $\sum_{i \in I} n_i \xi_i$ of (possibly infinitely many) irreducible characters ξ_i.

The paper is organized as follows. The first two sections are preliminary, collecting the necessary category-theoretic background. The Ziegler spectrum is defined in the third section where its main properties are described. The remaining sections are devoted to the Ziegler spectrum of a ring R. Examples are given in the fourth section. Duality is treated in the fifth section. In the sixth section, we briefly discuss finite matrix subgroups. In the following section, examples of Serre subcategories are given and we show how they are used in the analysis of R-modules. In the final two sections, we study the Grothendieck group $K_0(\text{coh-}\mathscr{C})$ and its characters.

Throughout this article, R will denote an associative ring with identity. By the unqualified term R-module is meant a unital left R-module. The category of R-modules is denoted by R-Mod; the category of right (unital) R-modules by Mod-R. If R is the ring of integers, then R-Mod is abbreviated to Ab. The full subcategory of R-Mod (Mod-R) of the finitely presented (right) R-modules is denoted by R-mod (mod-R).

Throughout this article, $\mathscr{C}$ will denote a Grothendieck category. By that we mean that $\mathscr{C}$ is an abelian category with a generator, that colimits exist in $\mathscr{C}$ and that direct limits are exact. We shall freely invoke the fact [**34**, Corollary X.4.3] that every object $X \in \mathscr{C}$ has an injective envelope $E(X) \in \mathscr{C}$.

If $\mathscr{B}$ is a category, then by a subcategory $\mathscr{A}$ of $\mathscr{B}$ we shall always mean a *full* subcategory of $\mathscr{B}$. For concepts such as subobject, epimorphism, injectivity, etc. we shall use the prefix $\mathscr{A}$-subobject or $\mathscr{B}$-subobject to indicate the context. This prefix may be omitted if the concept in question is absolute with respect to the inclusion $\mathscr{A} \subseteq \mathscr{B}$. To indicate the context of an operation, for example Ker η, $E(X)$ or $\varinjlim X_i$, we shall use a subscript, for example, $\text{Ker}_{\mathscr{A}}\,\eta$, $E_{\mathscr{A}}(X)$, etc. which may also be omitted in case of absoluteness.

The principal section of the paper is the third, in which the Ziegler spectrum is defined. Most of the results are categorical variants of results of Ziegler [**37**]. The present point of view, which stresses Serre subcategories is best encapsulated by Theorem 3.8 which adapts a result [**26**, Theorem 3.3] of Prest to the Ziegler spectrum. The model-theoretic variant of Theorem 3.8 was announced and proved in the Autumn of 1991 before a seminar at Brandeis University. I am grateful to the late Professor M. Auslander for giving me such an opportunity. His encouragement and generous contribution of ideas to this paper were a great inspiration. I also wish to thank W. W. Crawley-Boevey for many helpful suggestions.

1. *Preliminaries*

In this preliminary section, locally coherent categories are defined and we gather the information about such categories necessary for the sequel. To begin, we describe the subcategories of a Grothendieck category $\mathscr{C}$ which consist of the finitely generated objects, the finitely presented objects and the coherent objects respectively. These categories are ordered by the inclusions

$$\mathscr{C} \supseteq \text{fg-}\mathscr{C} \supseteq \text{fp-}\mathscr{C} \supseteq \text{coh-}\mathscr{C}.$$

Given objects $A, B \in \mathscr{C}$, the notation $A \leqslant B$ signifies that A is a subobject of B; a representative monomorphism μ will be denoted by $\mu\colon A \leqslant B$. For notation, we tend to adhere to the reference [**34**].

1.1. *Finitely presented objects*

An object $A \in \mathscr{C}$ is *finitely generated* if whenever there are subobjects $A_i \leqslant A$ for $i \in I$ satisfying

$$A = \sum_{i \in I} A_i,$$

then there is already a finite subset $J \subset I$ such that

$$A = \sum_{i \in J} A_i.$$

The subcategory of finitely generated objects is denoted by fg-$\mathscr{C}$. The category $\mathscr{C}$ is *locally*† *finitely generated* if every object $X \in \mathscr{C}$ is a directed sum

$$X = \sum_{i \in I} X_i$$

of finitely generated subobjects X_i. Clearly, the category R-Mod is locally finitely generated. All Grothendieck categories encountered in the sequel will be locally finitely generated.

A finitely generated object $B \in$ fg-$\mathscr{C}$ is *finitely presented* [**34**, § I.3] if every epimorphism $\eta\colon A \to B$ with A finitely generated has a finitely generated kernel Ker η. The subcategory of finitely presented objects of $\mathscr{C}$ is denoted by fp-$\mathscr{C}$. The respective categories of finitely presented R-modules are denoted by R-mod = fp-(R-Mod) and mod-R = fp-(Mod-R). The subcategory fp-$\mathscr{C}$ of $\mathscr{C}$ is closed under extensions, that is, if

$$0 \to X \to Y \to Z \to 0$$

is a short exact sequence in $\mathscr{C}$ with X and Z finitely presented, then Y is also finitely presented. If, on the other hand, the object Y is finitely presented, then Z is finitely presented if and only if X is finitely generated.

The most obvious example of a finitely presented object of $\mathscr{C}$ is a finitely generated projective object P. One says that $\mathscr{C}$ has *enough* finitely generated projectives if every finitely generated object $A \in \mathscr{C}$ admits an epimorphism $\eta\colon P \to A$ with P a finitely generated projective object. For example, the category R-Mod of left R-modules has enough finitely generated projectives. If $\mathscr{C}$ has enough finitely generated projectives, then by the remarks above, every finitely presented object $B \in \mathscr{C}$ is isomorphic to the cokernel of a morphism between finitely generated projective objects. This is expressed by an exact sequence

$$P_1 \to P_0 \to B \to 0$$

called a *projective presentation* of B. In the case of R-modules, these projectives may be taken free of finite rank.

The category $\mathscr{C}$ is *locally finitely presented* if every object $X \in \mathscr{C}$ is a direct limit

$$X = \varinjlim B_i$$

of finitely presented objects B_i. In such a category, every finitely generated object $A \in \mathscr{C}$ admits an epimorphism $\eta\colon B \to A$ from a finitely presented object B.

† If p is a property of an object of $\mathscr{C}$, then $\mathscr{C}$ is said to be locally p if there is a class of generators with the property p. As p varies, we shall opt for more operative definitions, but in each instance, the reader may check that the definition we give is equivalent to the standard.

PROPOSITION 1.1 [**23**, Appendice]. *If the Grothendieck category $\mathscr{C}$ is locally finitely generated with enough finitely generated projectives, then $\mathscr{C}$ is locally finitely presented.*

In particular, the category R-Mod of left R-modules is locally finitely presented.

PROPOSITION 1.2 [**34**, Proposition V.3.4]. *Let $\mathscr{C}$ be locally finitely generated. An object $B \in \mathscr{C}$ is finitely presented if and only if the functor*

$$\mathrm{Hom}_{\mathscr{C}}(B, -): \mathscr{C} \to \mathrm{Ab}$$

commutes with direct limits.

1.2. *Functor categories*

Let $\mathscr{B}$ be a small preadditive category. We denote by $(\mathscr{B}, \mathrm{Ab})$ (cf. [**34**, § IV.7]) the category whose objects are the additive functors $F: \mathscr{B} \to \mathrm{Ab}$ and whose morphisms are the natural transformations between functors. In this section we shall apply Proposition 1.1 to show that such a category is a locally finitely presented Grothendieck category. That it is Grothendieck follows from [**34**, Example V.2.2].

For $F, G \in (\mathscr{B}, \mathrm{Ab})$, we say that F is subfunctor of G, or $F \subseteq G$, if for each $X \in \mathscr{B}$, there is given an inclusion $F(X) \subseteq G(X)$ of abelian groups and whenever $f: X \to Y$ is a $\mathscr{B}$-morphism, then $F(f) = G(f)|_{F(X)}$. For example, if $\alpha: F \to G$ is a $(\mathscr{B}, \mathrm{Ab})$-morphism, then the kernel of α, defined for each $X \in \mathscr{B}$ by

$$(\mathrm{Ker}\,\alpha)(X) := \mathrm{Ker}\,\alpha_X,$$

is a subfunctor of F. Similarly, the image of α, defined by

$$(\mathrm{Im}\,\alpha)(X) := \mathrm{Im}\,\alpha_X,$$

is a subfunctor of G. Finally, the cokernel of α is defined as the quotient functor $\mathrm{Coker}\,\alpha := G/\mathrm{Im}\,\alpha$. It is readily verified that the diagram

$$F \xrightarrow{\alpha} G \xrightarrow{\beta} H$$

of $(\mathscr{B}, \mathrm{Ab})$-morphisms is exact if and only if $\mathrm{Im}\,\alpha = \mathrm{Ker}\,\beta$, that is, if and only if for each $X \in \mathscr{B}$, the sequence of Ab-morphisms

$$F(X) \xrightarrow{\alpha_X} G(X) \xrightarrow{\beta_X} H(X)$$

is exact.

A functor $F \in (\mathscr{B}, \mathrm{Ab})$ is called *representable* if it is isomorphic to one of the functors

$$(X, -) := \mathrm{Hom}_{\mathscr{B}}(X, -)$$

where $X \in \mathscr{B}$. Every representable functor is an example of a finitely generated functor. For, suppose that

$$(X, -) = \sum_{i \in I} F_i.$$

There is a finite subset J of I such that $1_X \in (\sum_{i\in J} F_i)(X) = \sum_{i\in J} F_i(X) \subseteq (X, X)$. It is easy to check that then $(X, -) = \sum_{i\in J} F_i$.

YONEDA'S LEMMA. *Let $X \in \mathcal{B}$ and $F \in (\mathcal{B}, \mathrm{Ab})$. There is an isomorphism of abelian groups*

$$\Theta_{X,F}\colon \operatorname{Hom}_{(\mathcal{B},\mathrm{Ab})}[(X,-),F] \to F(X)$$

defined by $\Theta_{X,F}(\eta) = \eta_X(1_X)$ which is natural in both X and F.

It is immediate from Yoneda's Lemma that the functor $X \mapsto (X, -)$ is full, that is, that every $(\mathcal{B}, \mathrm{Ab})$-morphism $\eta\colon (X, -) \to (Y, -)$ is of the form $\eta = (f, -)$ for some $\mathcal{B}$-morphism $f\colon Y \to X$.

Another consequence is that every representable functor is projective. For, let

$$0 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 0$$

be a short exact sequence in $(\mathcal{B}, \mathrm{Ab})$. Applying the functor $\operatorname{Hom}_{(\mathcal{B},\mathrm{Ab})}[(X,-), ?]$ gives a sequence which, by Yoneda's Lemma, is isomorphic to the exact sequence

$$0 \longrightarrow F(X) \xrightarrow{\alpha_X} G(X) \xrightarrow{\beta_X} H(X) \longrightarrow 0.$$

PROPOSITION 1.3 [**34**, Corollary IV.7.5]. *Let $\mathcal{B}$ be a small preadditive category. The functor category $(\mathcal{B}, \mathrm{Ab})$ is locally finitely generated with enough finitely generated projectives. Therefore $(\mathcal{B}, \mathrm{Ab})$ is a locally finitely presented Grothendieck category.*

Proof. First, we will note that $(\mathcal{B}, \mathrm{Ab})$ is locally finitely generated. Let $F \in (\mathcal{B}, \mathrm{Ab})$. For every $X \in \mathcal{B}$ and $x \in F(X)$, there is, by Yoneda's Lemma, a morphism $\eta = \Theta^{-1}_{X,F}(x)\colon (X, -) \to F$ such that $x = \eta_X(1_X)$. As $\mathcal{B}$ is small,

$$F = \sum_{X\in\mathcal{B}} \Big(\sum_{x\in F(X)} \operatorname{Im} \Theta^{-1}_{X,F}(x) \Big)$$

and each $\operatorname{Im} \Theta^{-1}_{X,F}(x)$, a quotient functor of $(X, -)$, is finitely generated. If F is finitely generated, then F is already the sum of finitely many of the factors and is therefore a quotient functor of a finite coproduct of representable functors. But then it is a quotient of a finitely generated projective object.

1.3. *Coherent objects*

A subcategory $\mathcal{A} \subseteq \mathcal{C}$ is *exact* if it is abelian and the inclusion functor of $\mathcal{A}$ into $\mathcal{C}$ is exact.

PROPOSITION 1.4 [**11**, Theorem 3.41]. *A subcategory $\mathcal{A}$ of $\mathcal{C}$ is an exact subcategory if and only if the following two conditions hold.*

(1) *If $A_1, A_2 \in \mathcal{A}$ then the coproduct $A_1 \amalg A_2$ is an object of $\mathcal{A}$.*

(2) *If $\eta\colon A_1 \to A_2$ is a morphism in $\mathcal{A}$, then both the $\mathcal{C}$-kernel and $\mathcal{C}$-cokernel of η are objects of $\mathcal{A}$.*

Since the subcategory fp-$\mathcal{C}$ of $\mathcal{C}$ is closed under extensions, Condition (1) of

Proposition 1.4 is satisfied. And if $\eta\colon B_1 \to B_2$ is a morphism in fp-$\mathscr{C}$, then Coker η is also finitely presented. However, the kernel Ker η is not necessarily finitely presented. Because the category fp-$\mathscr{C}$ is not always an exact subcategory of $\mathscr{C}$, we will restrict our attention to a smaller category which *is* exact.

A finitely presented object $C \in \mathscr{C}$ is *coherent* if every finitely generated subobject $B \leqslant C$ is finitely presented. Equivalently, every epimorphism $\eta\colon C \to A$ with A finitely presented has a finitely presented kernel. Evidently, a finitely generated subobject of a coherent object is also coherent. The subcategory of coherent objects of $\mathscr{C}$ is denoted by coh-$\mathscr{C}$.

PROPOSITION 1.5 [**1**, p. 199]. *The category* coh-$\mathscr{C}$ *is an exact subcategory of* $\mathscr{C}$ *closed under extensions.*

Proof. Let $\eta\colon C_1 \to C_2$ be a morphism in coh-$\mathscr{C}$. Since Im η is a finitely generated subobject of C_2, it is finitely presented. Then Ker η is a finitely generated subobject of C_1 and is therefore coherent. To check that Coker $\eta \cong C_2/\mathrm{Im}\,\eta$ is coherent, let Y be a finitely generated subobject of $C_2/\mathrm{Im}\,\eta$. Its preimage in C_2 is a finitely generated subobject Y_0 of C_2 containing Im η. By the hypothesis, Y_0 is then finitely presented and therefore so is $Y \cong Y_0/\mathrm{Im}\,\eta$.

Now we verify that coh-$\mathscr{C}$ is closed under extensions. Let

$$0 \longrightarrow C_1 \xrightarrow{\alpha} Y \xrightarrow{\beta} C_2 \longrightarrow 0$$

be a short exact sequence with C_1 and C_2 coherent. If $X \leqslant Y$ is finitely generated, we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & X & \longrightarrow & A_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_1 & \underset{\alpha}{\longrightarrow} & Y & \underset{\beta}{\longrightarrow} & C_2 & \longrightarrow & 0 \end{array}$$

with exact rows and each of the vertical morphisms is monic ($A_2 = \mathrm{Im}\,[\beta|_X]$). Now A_2 is a finitely generated subobject of the coherent object C_2 and is therefore finitely presented. But since X is finitely generated, so is A_1. Now A_1 is a finitely generated subobject of the coherent object C_1 and is therefore finitely presented. As fp-$\mathscr{C}$ is closed under extensions, $X \in$ fp-$\mathscr{C}$.

The Grothendieck category $\mathscr{C}$ is *locally coherent* if every object of $\mathscr{C}$ is a direct limit of coherent objects.

THEOREM 1.6 [**31**, § 2]. *The following conditions on a locally finitely presented Grothendieck category* $\mathscr{C}$ *are equivalent:*

(1) $\mathscr{C}$ *is locally coherent;*

(2) fp-$\mathscr{C}$ = coh-$\mathscr{C}$;

(3) fp-$\mathscr{C}$ *is an exact subcategory of* $\mathscr{C}$;

(4) fp-$\mathscr{C}$ *is an abelian category.*

Proof. It is clear from Proposition 1.5 that Condition (2) implies the others. To see that (1) $\Rightarrow$ (2), note that if $\mathscr{C}$ is locally coherent, then every finitely presented B is the quotient of a coherent object and is therefore coherent. Thus Conditions (1) and (2) are equivalent. The implication (3) $\Rightarrow$ (4) is just part of the definition of an exact subcategory, so it remains to prove that (4) $\Rightarrow$ (2). Let $B \in$ fp-$\mathscr{C}$ and let $A \leq B$ be finitely generated. The quotient map π: $B \to B/A$ is a morphism in the category fp-$\mathscr{C}$. Let κ: $A' \to B$ be the (fp-$\mathscr{C}$)-kernel of π. We shall prove that κ is the $\mathscr{C}$-kernel of π. Then κ is a $\mathscr{C}$-monomorphism and $A \cong A' \in$ fp-$\mathscr{C}$.

Suppose that γ: $X \to B$ satisfies $\pi\gamma = 0$ and write $X = \varinjlim X_i$ as a direct limit of finitely presented objects X_i with the compatible family of morphisms α_i: $X_i \to X$. Each $\gamma\alpha_i$ is a morphism in fp-$\mathscr{C}$ and thus factors uniquely through κ. But then γ factors through κ uniquely.

Corollary 1.7 [**23**, Appendice]. *If the Grothendieck category $\mathscr{C}$ is locally finitely generated with enough finitely generated coherent projectives, then $\mathscr{C}$ is locally coherent.*

Proof. Because coh-$\mathscr{C}$ is abelian and every finitely presented object is the cokernel of a morphism between finitely generated projective objects, every finitely presented object is coherent.

For example, a ring R is *left coherent* if every finitely generated left ideal $I \subseteq {}_RR$ is a finitely presented left R-module. In other words, the object ${}_RR$ of R-Mod is coherent. Consequently, every finitely generated projective left R-module is coherent and so, by the corollary, R-Mod is locally coherent.

2. *Examples of locally coherent categories*

In this section, we present some examples of locally coherent Grothendieck categories and verify that they are indeed such. To begin we note that many functor categories are locally coherent Grothendieck categories and it is instructive for the reader to keep these examples in mind throughout the sequel. Let $\mathscr{C}$ denote a locally coherent Grothendieck category and $\mathscr{S}$ a Serre subcategory (defined below) of coh-$\mathscr{C}$. In each of the two subsections a method is given of obtaining from $\mathscr{S}$ a subcategory of $\mathscr{C}$ that is also a locally coherent Grothendieck category. All the results in this section are classical, often true in greater generality (cf. [**12, 25**]), but our pedestrian approach is intended to lend concreteness to the categories of the title. Throughout this section as well as the sequel, $\mathscr{C}$ will denote a locally coherent Grothendieck category.

Proposition 2.1 [**1**, Theorem 2.2.b; **34**, Corollary IV.7.5]. *Let $\mathscr{B}$ be a small additive category, that is, $\mathscr{B}$ is preadditive, has finite products/coproducts and idempotents split in $\mathscr{B}$. Then every finitely generated projective object in* ($\mathscr{B}$, Ab) *is representable.* If $\mathscr{B}$ *has cokernels, then* ($\mathscr{B}$, Ab) *is locally coherent and* coh-($\mathscr{B}$, Ab) *has projective global dimension at most* 2.

Proof. By the proof of Proposition 1.3, every finitely generated projective object $P \in (\mathcal{B}, \mathrm{Ab})$ is a coproduct factor of a finite coproduct of representable objects $\coprod_{i=1}^{n} (B_i, -) \cong (\coprod_{i=1}^{n} B_i, -)$. This coproduct factor P of $(\coprod_{i=1}^{n} B_i, -)$ corresponds to an idempotent in

$$\mathrm{End}_{(\mathcal{B},\mathrm{Ab})}\Big(\coprod_{i=1}^{n} B_i, -\Big) \cong \mathrm{End}_{\mathcal{B}}\Big(\coprod_{i=1}^{n} B_i\Big).$$

As idempotents split in $\mathcal{B}$, this corresponds to a coproduct factor B of $\coprod_{i=1}^{n} B_i$ which has the property that $(B, -) \cong P$.

By Proposition 1.7, it suffices to show that every finitely generated projective object, that is, every representable object of $(\mathcal{B}, \mathrm{Ab})$, is coherent. So let $A \in \mathcal{B}$ and let $X \subseteq (A, -)$ be a finitely generated subfunctor. An epimorphism η: $(B, -) \to X$ lifts to a morphism η: $(B, -) \to (A, -)$ which, by Yoneda's Lemma, has the form $\eta = (f, -)$ for some $\mathcal{B}$-morphism f: $A \to B$. By hypothesis, $C = \mathrm{Coker}\, f \in \mathcal{B}$ and the exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\mathcal{B}$ induces an exact sequence

$$0 \longrightarrow (C, -) \xrightarrow{(g,-)} (B, -) \xrightarrow{(f,-)} (A, -) \longrightarrow (A, -)/X \longrightarrow 0$$

in $(\mathcal{B}, \mathrm{Ab})$ which gives a projective presentation of $X \cong \mathrm{Im}(f, -)$. If $F \in (\mathcal{B}, \mathrm{Ab})$ is coherent, then it is isomorphic to a functor of the form $(A, -)/X$ as above and so has projective resolution of length at most 2.

Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then fp-$\mathcal{A}$ is an additive category with cokernels. By Proposition 2.1, the functor category (fp-$\mathcal{A}$, Ab) is locally coherent. For example, if R is a ring then the functor category

$${}_R\mathcal{C} = (\text{mod-}R, \mathrm{Ab})$$

is a locally coherent Grothendieck category. This category ${}_R\mathcal{C}$ is called the category of generalized left R-modules on account of the fully faithful right exact functor $- \otimes_R ?$: R-Mod $\to {}_R\mathcal{C}$ defined by the rule ${}_RM \mapsto - \otimes_R M$. If ${}_RM$ is a finitely presented left R-module, it is proved in [**1**, Lemma 6.1] that the functor $- \otimes_R M$ is a coherent object of ${}_RC$. To see this, consider a presentation of ${}_RM$ by finitely generated free modules

$${}_RR^{(m)} \longrightarrow {}_RR^{(n)} \longrightarrow {}_RM \longrightarrow 0.$$

As the tensor functor is right exact, this gives an exact sequence in ${}_R\mathcal{C}$,

$$- \otimes_R R^{(m)} \to - \otimes_R R^{(n)} \to - \otimes_R M \to 0,$$

which is a presentation of $- \otimes_R M$ in ${}_R\mathcal{C}$ by finitely generated projective objects (since $- \otimes_R R^{(n)} \cong (- \otimes_R R)^{(n)} \cong (R, -)^{(n)}$).

If $\mathcal{C}$ is a locally coherent Grothendieck category, then coh-$\mathcal{C}$ and hence $(\text{coh-}\mathcal{C})^{\mathrm{op}}$ are abelian categories. By Proposition 2.1, the functor category $((\text{coh-}\mathcal{C})^{\mathrm{op}}, \mathrm{Ab})$ is also a locally coherent Grothendieck category. For example, if

R is a left coherent ring, then the functor category $((R\text{-mod})^{\mathrm{op}}, \mathrm{Ab})$ is a locally coherent Grothendieck category.

2.1. *Hereditary torsion subcategories of finite type*

A subcategory $\mathcal{T} \subseteq \mathcal{C}$ is a *torsion* subcategory [**34**, Chapter VI] if it is closed under quotient objects, extensions and coproducts. The torsion subcategory $\mathcal{T} \subseteq \mathcal{C}$ is *hereditary* if, in addition, it is closed under subobjects. If $\mathcal{A} \subseteq \mathcal{C}$ is an arbitrary subcategory, we denote by $\mathcal{T}(\mathcal{A})$ the smallest hereditary torsion subcategory of $\mathcal{C}$ to contain $\mathcal{A}$.

Let $\mathcal{T} \subseteq \mathcal{C}$ be a hereditary torsion subcategory. It is clear that an object $T \in \mathcal{T}$ is $\mathcal{C}$-finitely generated if and only if it is $\mathcal{T}$-finitely generated. Hence the equation

$$\text{fg-}\mathcal{T} = \mathcal{T} \cap \text{fg-}\mathcal{C}.$$

From the definition of a finitely presented object, it follows that $\text{fp-}\mathcal{T} \supseteq \mathcal{T} \cap \text{fp-}\mathcal{C}$. As we are assuming that $\mathcal{C}$ is locally coherent, we have that $\text{fp-}\mathcal{C} = \text{coh-}\mathcal{C}$ and a similar argument yields the inclusion

$$\text{coh-}\mathcal{T} \supseteq \mathcal{T} \cap \text{coh-}\mathcal{C}.$$

In this subsection, we consider hereditary torsion subcategories $\mathcal{T}$ of $\mathcal{C}$ that are of *finite type,* meaning that they have the form $\mathcal{T} = \mathcal{T}(\mathcal{A})$ where $\mathcal{A}$ consists of coherent objects. We may write $\mathcal{T} = \mathcal{T}(\mathcal{S})$ where $\mathcal{S} = \mathcal{T} \cap \text{coh-}\mathcal{C}$. Evidently, such a subcategory $\mathcal{S}$ is a *Serre subcategory* of coh-$\mathcal{C}$, that is, if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence in coh-$\mathcal{C}$, then $B \in \mathcal{S}$ if and only if $A, C \in \mathcal{S}$. Thus a hereditary torsion subcategory of finite type has the general form $\mathcal{T}(\mathcal{S})$ where $\mathcal{S}$ is a Serre subcategory of coh-$\mathcal{C}$.

Serre subcategories $\mathcal{S}$ of coh-$\mathcal{C}$ arise in the following natural way. An object $M \in \mathcal{C}$ is *coh-injective* if $\mathrm{Ext}^1_{\mathcal{C}}(C, M) = 0$ for each $C \in \text{coh-}\mathcal{C}$. Then the subcategory

$$\mathcal{S}(M) = \{C \in \text{coh-}\mathcal{C}\colon\ (C, M) = 0\}$$

is Serre. For, let $0 \to A \to B \to C \to 0$ be a short exact sequence in coh-$\mathcal{C}$. Applying the functor $(-, M)$ gives an exact sequence

$$0 \to (C, M) \to (B, M) \to (A, M) \to \mathrm{Ext}^1_{\mathcal{C}}(C, M) = 0$$

which shows that $B \in \mathcal{S}(M)$ if and only if $A, C \in \mathcal{S}(M)$. We shall note later (Corollary 3.11) that every Serre subcategory of coh-$\mathcal{C}$ arises in this fashion.

Proposition 2.2 [**19**, Theorem B.15]. *Let $\mathcal{B}$ be a small additive category with cokernels. An object $M \in (\mathcal{B}, \mathrm{Ab})$ is coh-injective if and only if it is right exact.*

Proof. Let $F \in \text{coh-}(\mathcal{B}, \mathrm{Ab})$ with a projective presentation

$$(B, -) \xrightarrow{(f,-)} (A, -) \longrightarrow F \longrightarrow 0.$$

As in the proof of Proposition 2.1, this may be extended to a projective resolution of F with $C = \operatorname{Coker} f$,

$$0 \longrightarrow (C,-) \longrightarrow (B,-) \xrightarrow{(f,-)} (A,-) \longrightarrow F \longrightarrow 0.$$

If $M \in (\mathscr{B}, \mathrm{Ab})$ is coh-injective, then the sequence

$$((A,-),M) \xrightarrow{((f,-),M)} ((B,-),M) \longrightarrow ((C,-),M) \longrightarrow 0$$

is exact. By Yoneda's Lemma, this sequence is isomorphic to the sequence

$$M(A) \to M(B) \to M(C) \to 0$$

yielding the right exactness of M. Conversely, if M is right exact, retracing this argument shows that $\operatorname{Ext}^1(F, M) = 0$ for any coherent object $F \in (\mathscr{B}, \mathrm{Ab})$.

An argument as in [**34**, Proposition IV.10.1] shows that every right exact functor M in ${}_R\mathscr{C} = (\text{mod-}R, \mathrm{Ab})$ is of the form $M \cong - \otimes_R M(R_R)$. Thus the category R-Mod of left R-modules is recovered as the subcategory of coh-injective objects of the category ${}_R\mathscr{C}$.

Let $\mathscr{S} \subseteq \text{coh-}\mathscr{C}$ be a Serre subcategory. Denote by $\vec{\mathscr{S}}$ the subcategory of $\mathscr{C}$ which consists of direct limits of objects in $\mathscr{S}$. We shall show that $\vec{\mathscr{S}} = \mathscr{T}(\mathscr{S})$. One direction is easily seen. Every direct limit $\varinjlim S_i$ is the quotient of a coproduct of objects from $\mathscr{S}$ and therefore lies in $\mathscr{T}(\mathscr{S})$. Indeed, an argument as in [**23**, Appendice] shows that the objects of $\vec{\mathscr{S}}$ are precisely those objects X of $\mathscr{C}$ which admit an epimorphism $\eta\colon \coprod_{i \in I} S_i \to X$ from a coproduct of objects in $\mathscr{S}$.

Proposition 2.3. *The following are equivalent for a finitely generated object* $A \in \mathscr{C}$:

(1) $A \in \vec{\mathscr{S}}$;

(2) *there is an epimorphism* $\eta\colon S \to A$ *with* $S \in \mathscr{S}$;

(3) *if* $B \in \text{coh-}\mathscr{C}$ *and* $\varepsilon\colon B \to A$ *is an epimorphism, then* ε *factors through a quotient* S *of* B *which lies in* $\mathscr{S}$.

Proof. The equivalence of Conditions (1) and (2) is clear from the remarks above together with the fact that $\mathscr{S}$ is closed under finite coproducts. Evidently, Condition (3) implies Condition (2). So assume Condition (2) with intent to prove Condition (3). Write $\operatorname{Ker} \eta = \sum_{i \in I} S_i$ as a directed union of finitely generated subobjects of S. By the exactness of direct limit functors, $A \cong \varinjlim (S/S_i)$ and therefore by Proposition 1.2, any epimorphism $\varepsilon\colon B \to A$ will factor through one of the finitely presented objects S/S_i of $\mathscr{S}$ as in the following diagram

$$\begin{array}{ccc} B & \xrightarrow{\alpha} & S/S_i \\ {\scriptstyle\varepsilon}\downarrow & \swarrow & \\ A & & \end{array}$$

But then ε factors through the coherent quotient $\operatorname{Im} \alpha \in \mathscr{S}$.

The characterization of $\vec{\mathcal{S}}$ mentioned above ensures that $\vec{\mathcal{S}}$ is closed under coproducts, quotient objects and therefore, by [**34**, Proposition IV.8.4], colimits. So to check that $Y \in \vec{\mathcal{S}}$ it suffices to verify the same for every finitely generated subobject A of Y.

PROPOSITION 2.4. *The subcategory $\vec{\mathcal{S}}$ of $\mathcal{C}$ is closed under subobjects. An object $X \in \mathcal{C}$ is in $\vec{\mathcal{S}}$ if and only if any morphism α: $B \to X$ with $B \in$ coh-$\mathcal{C}$ factors through an object $S \in \mathcal{S}$.*

Proof. Suppose $A \leq \varinjlim S_i$ is finitely generated. It is enough to show that $A \in \vec{\mathcal{S}}$. If $B \in \mathcal{C}$ is coherent with an epimorphism ε: $B \to A$, then by Proposition 1.2, ε factors through one of the S_i. By Proposition 2.3, $A \in \vec{\mathcal{S}}$. To prove the second statement, apply Proposition 2.3 to Im α.

THEOREM 2.5. *Let $\mathcal{T} \subseteq \mathcal{C}$ be a hereditary torsion subcategory of finite type. If $\mathcal{S}$ is the Serre subcategory $\mathcal{T} \cap$ coh-$\mathcal{C}$ of coh-$\mathcal{C}$, then $\mathcal{T} = \vec{\mathcal{S}}$.*

Proof. It remains to be seen that $\vec{\mathcal{S}}$ is closed under extensions. So let

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{\pi} Y/X \longrightarrow 0$$

be a short exact sequence in $\mathcal{C}$ such that $X, Y/X \in \vec{\mathcal{S}}$ and consider a morphism α: $B \to Y$ with $B \in$ coh-$\mathcal{C}$. We must prove that α factors through an object in $\mathcal{S}$. We know that $\pi\alpha$ factors through a quotient B/A in $\mathcal{S}$. The finitely generated subobject $A \leq B$ is then coherent and we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & B/A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \underset{\pi}{\longrightarrow} & Y/X & \longrightarrow & 0 \end{array}$$

with exact rows. As $X \in \vec{\mathcal{S}}$, β factors through a quotient $A/K \in \mathcal{S}$ where $K \leq \operatorname{Ker} \beta \leq \operatorname{Ker} \alpha$. Thus α factors through the quotient object B/K. Now $K \leq A \leq B$ with B/A and A/K in $\mathcal{S}$. As $\mathcal{S}$ is closed under extensions, $B/K \in \mathcal{S}$ and therefore $Y \in \vec{\mathcal{S}}$.

COROLLARY 2.6. *Let $\mathcal{T} \subseteq \mathcal{C}$ be a hereditary torsion subcategory of finite type. Then*

$$\text{coh-}\mathcal{T} = \mathcal{T} \cap \text{coh-}\mathcal{C}.$$

Proof. Let $A \in$ coh-$\mathcal{T}$. By Theorem 2.5 and Proposition 2.3 there is an epimorphism η: $S \to A$ with $S \in \mathcal{T} \cap$ coh-$\mathcal{C} \subseteq$ coh-$\mathcal{T}$. Now Ker η is $\mathcal{T}$-finitely generated and hence $\mathcal{C}$-finitely generated and therefore $\mathcal{C}$-coherent. But then A is $\mathcal{C}$-coherent.

COROLLARY 2.7. *Let $\mathcal{S} \subseteq$ coh-$\mathcal{C}$ be a Serre subcategory. The hereditary torsion subcategory $\mathcal{T}(\mathcal{S})$ of finite type is a locally coherent Grothendieck category.*

Proof. We have noted already how $\mathcal{T}(\mathcal{S})$ is closed under $\mathcal{C}$-colimits. Since these are also $\mathcal{T}(\mathcal{S})$-colimits, $\mathcal{T}(\mathcal{S})$ is closed under such limits. As $\mathcal{T}(\mathcal{S})$ is an exact subcategory, direct limits are exact in $\mathcal{T}(\mathcal{S})$ and by the definition of $\vec{\mathcal{S}}$ every object in $\mathcal{T}(\mathcal{S})$ is a direct limit of $\mathcal{T}$-coherent objects. The coproduct $\coprod_{S \in \mathcal{S}} S$ is a generator of $\mathcal{T}$.

For example, the category Ab of abelian groups is a locally coherent Grothendieck category and the torsion groups form a hereditary torsion subcategory Tors of finite type. Thus Tors is a locally coherent Grothendieck category. Furthermore (cf. [**34**, Example IV.4.3]), Tors has no non-zero projective objects.

THEOREM 2.8. *There is an inclusion-preserving bijective correspondence between Serre subcategories $\mathcal{S}$ of* coh-$\mathcal{C}$ *and hereditary torsion subcategories $\mathcal{T}$ of $\mathcal{C}$ of finite type. This correspondence is given by the functions*

$$\mathcal{S} \mapsto \mathcal{T}(\mathcal{S}) = \vec{\mathcal{S}},$$
$$\mathcal{T} \mapsto \text{coh-}\mathcal{T} = \mathcal{T} \cap \text{coh-}\mathcal{C},$$

which are mutual inverses.

Let $\vec{\mathcal{S}}$ be a hereditary torsion subcategory of $\mathcal{C}$ of finite type. The corresponding torsion functor is denoted by

$$t_{\mathcal{S}}\colon\ \mathcal{C} \to \vec{\mathcal{S}}.$$

This functor assigns to an object $X \in \mathcal{C}$ the maximal subobject $t_{\mathcal{S}}(X) \leqslant X$ from $\vec{\mathcal{S}}$. The subobject $t_{\mathcal{S}}(X)$ is unique by the properties of a torsion subcategory. If $X \in \vec{\mathcal{S}}$ and $Y \in \mathcal{C}$, then there is an isomorphism

$$\mathrm{Hom}_{\vec{\mathcal{S}}}(X, t_{\mathcal{S}}(Y)) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

natural in both X and Y. This is because every morphism $\eta\colon X \to Y$ in $\mathcal{C}$ with $X \in \vec{\mathcal{S}}$ has the property that $\mathrm{Im}\, \eta \leqslant t_{\mathcal{S}}(Y)$. In short, the torsion functor $t_{\mathcal{S}}$ is the right adjoint of the inclusion functor $\vec{\mathcal{S}} \subseteq \mathcal{C}$.

2.2. *Localization*

Throughout this section, $\vec{\mathcal{S}} \subseteq \mathcal{C}$ will denote a hereditary torsion subcategory of finite type and $t = t_{\mathcal{S}}$ the corresponding torsion functor. We shall describe in this section another category $\mathcal{C}/\vec{\mathcal{S}}$ of $\mathcal{C}$ which is also a locally coherent Grothendieck category.

DEFINITION. An object $X \in \mathcal{C}$ is *$\vec{\mathcal{S}}$-torsion-free* if $t(X) = 0$. Let t^1 denote the first higher derived functor of the left exact functor t. An $\vec{\mathcal{S}}$-torsion-free object $X \in \mathcal{C}$ is *$\vec{\mathcal{S}}$-closed* if $t^1(X) = 0$. The subcategory of $\vec{\mathcal{S}}$-closed objects of $\mathcal{C}$ is denoted by $\mathcal{C}/\vec{\mathcal{S}}$. This category $\mathcal{C}/\vec{\mathcal{S}}$ is called the *quotient category* of $\mathcal{C}$ by $\vec{\mathcal{S}}$.

If $E \in \mathcal{C}$ is an injective object, then $t^1(E) = 0$, so E is $\vec{\mathcal{S}}$-closed if and only if it is $\vec{\mathcal{S}}$-torsion-free. Let X be $\vec{\mathcal{S}}$-torsion-free. Because the torsion theory is hereditary, the injective envelope $E(X)$ is also $\vec{\mathcal{S}}$-torsion-free. The short exact sequence

$$0 \to X \to E(X) \to E(X)/X \to 0$$

gives rise to a long exact sequence

$$0 \to t(X) \to t(E(X)) = 0 \to t(E(X)/X) \to t^1(X) \to t^1(E(X)) = 0$$

showing that $t^1(X) \cong t(E(X)/X)$.

PROPOSITION 2.9. *Let $X \in \mathscr{C}$ be $\vec{\mathscr{S}}$-torsion-free. Then X is $\vec{\mathscr{S}}$-closed if and only if there is no $\vec{\mathscr{S}}$-torsion-free (essential) extension $Y \geq X$ with $Y/X \in \vec{\mathscr{S}}$.*

Proof. The proposition follows immediately from the remarks above once we show that an $\vec{\mathscr{S}}$-torsion-free extension $Y \geq X$ with $Y/X \in \vec{\mathscr{S}}$ must be essential. But if $A \leq Y$ and $A \cap X = 0$, then $A \in \vec{\mathscr{S}}$ and therefore $A = 0$.

Given any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathscr{C}$, there is a long exact sequence in $\vec{\mathscr{S}}$,

$$0 \to t(X) \to t(Y) \to t(Z) \to t^1(X) \to t^1(Y) \to t^1(Z).$$

From this long exact sequence, we see that when Z is $\vec{\mathscr{S}}$-closed, then X is $\vec{\mathscr{S}}$-closed if and only if Y is.

PROPOSITION 2.10. *If α: $Y \to W$ is a morphism in $\mathscr{C}/\vec{\mathscr{S}}$, then the $\mathscr{C}$-kernel of α is $\vec{\mathscr{S}}$-closed.*

Proof. Let $X = \operatorname{Ker}_{\mathscr{C}} \alpha$ and $Z = \operatorname{Im}_{\mathscr{C}} \alpha$. Both X and Z are $\vec{\mathscr{S}}$-torsion-free and we have α as part of a short exact sequence

$$0 \longrightarrow X \longrightarrow Y \xrightarrow{\ \alpha\ } Z \longrightarrow 0.$$

By the long exact sequence and the hypothesis $t^1(Y) = 0$, it follows that $t^1(X) = 0$ and that $X \in \mathscr{C}/\vec{\mathscr{S}}$.

In particular, a $\mathscr{C}/\vec{\mathscr{S}}$-morphism is a monomorphism if and only if it is a $\mathscr{C}$-monomorphism. So for $A, B \in \mathscr{C}/\vec{\mathscr{S}}$ the relation $A \leq B$ holds in $\mathscr{C}/\vec{\mathscr{S}}$ if and only if it holds in $\mathscr{C}$.

DEFINITION. Let $X \in \mathscr{C}$. A localization of X at $\vec{\mathscr{S}}$ is a morphism λ_X: $X \to X_{\mathscr{S}}$ such that $X_{\mathscr{S}}$ is $\vec{\mathscr{S}}$-closed satisfying the condition that given any morphism α: $X \to W$ with W an $\vec{\mathscr{S}}$-closed object, there is a unique morphism $\alpha_{\mathscr{S}}$: $X_{\mathscr{S}} \to W$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\lambda_X} & X_{\mathscr{S}} \\ {\scriptstyle \alpha}\downarrow & \swarrow{\scriptstyle \alpha_{\mathscr{S}}} & \\ W & & \end{array}$$

is commutative.

It is clear from the definition that a localization of X at $\vec{\mathscr{S}}$ is unique up to isomorphism. If $X \in \mathcal{C}$ is $\vec{\mathcal{S}}$-torsion-free, then the localization of X at $\vec{\mathcal{S}}$ is constructed as follows. Let $X_{\mathcal{S}} \geqslant X$ be a maximal essential extension with respect to the condition $X_{\mathcal{S}}/X \in \vec{\mathcal{S}}$. Such an $X_{\mathcal{S}}$ is clearly $\vec{\mathcal{S}}$-torsion-free and by Proposition 2.9, it is $\vec{\mathcal{S}}$-closed. If α: $X \to W$ is a morphism with W an $\vec{\mathcal{S}}$-closed object, we get a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & X_{\mathcal{S}} & \longrightarrow & X_{\mathcal{S}}/X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_{\mathcal{S}}} & & \downarrow{\scriptstyle \alpha_{\mathcal{S}}/\alpha} & & \\ 0 & \longrightarrow & W & \longrightarrow & E(W) & \longrightarrow & E(W)/W & \longrightarrow & 0 \end{array}$$

The morphism $\alpha_{\mathcal{S}}/\alpha = 0$ because $X_{\mathcal{S}}/X \in \vec{\mathcal{S}}$ and $E(W)/W$ is $\vec{\mathcal{S}}$-torsion-free so that $\operatorname{Im} \alpha_{\mathcal{S}} \subseteq W$. The extension $\alpha_{\mathcal{S}}$ is unique. For, another $\alpha'_{\mathcal{S}}$ would induce a morphism $\alpha_{\mathcal{S}} - \alpha'_{\mathcal{S}}$: $X_{\mathcal{S}}/X \to W$ which must be zero since W is $\vec{\mathcal{S}}$-torsion-free. Thus the extension $X_{\mathcal{S}} \geqslant X$ is indeed the localization of X at $\vec{\mathcal{S}}$. In particular, if $X \leqslant W$ and W is $\vec{\mathcal{S}}$-closed, then $X_{\mathcal{S}} \leqslant W$.

To construct the localization at $\vec{\mathcal{S}}$ of a general object $X \in \mathcal{C}$, first apply the quotient map π: $X \to X/t(X)$. It is clear that any morphism from X to an $\vec{\mathcal{S}}$-closed object factors uniquely through π. Then it is easily seen that a localization of X at $\vec{\mathcal{S}}$ is obtained by composing this quotient map with the localization of the $\vec{\mathcal{S}}$-torsion-free object $X/t(X)$, that is, $\lambda_X = \lambda_{X/t(X)}\pi$. It follows that $X_{\mathcal{S}} = 0$ if and only if $X \in \vec{\mathcal{S}}$ and so for a coherent object C we have, by Theorem 2.8, that $C_{\mathcal{S}} = 0$ if and only if $C \in \mathcal{S}$. Because localization of X at $\vec{\mathcal{S}}$ is a solution to a universal problem, we have the following.

PROPOSITION 2.11. *If $X \in \mathcal{C}$ and $Y \in \mathcal{C}/\vec{\mathcal{S}}$, then there is an isomorphism*

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) \cong \operatorname{Hom}_{\mathcal{C}/\vec{\mathcal{S}}}(X_{\mathcal{S}}, Y)$$

natural both in X and Y. In words, the localization functor $(-)_{\mathcal{S}}$: $\mathcal{C} \to \mathcal{C}/\vec{\mathcal{S}}$ is the left adjoint of the inclusion functor $\mathcal{C}/\vec{\mathcal{S}} \subseteq \mathcal{C}$.

A subcategory $\mathcal{B} \subseteq \mathcal{C}$ is called *Giraud* [**34**, p. 214] if the inclusion functor has a left adjoint that preserves kernels. By Propositions 2.10 and 2.11, the subcategory $\mathcal{C}/\vec{\mathcal{S}}$ of $\mathcal{C}$ is Giraud. By [**34**, Propositions X.1.2 and X.1.3], every Giraud subcategory is a Grothendieck category. Because a left adjoint preserves colimits [**34**, Proposition IV.9.3], it preserves cokernels and the localization functor

$$(-)_{\mathcal{S}}\colon\ \mathcal{C} \to \mathcal{C}/\vec{\mathcal{S}}$$

is therefore exact. Another property [**34**, Proposition X.1.4] of a Giraud subcategory which we shall need is that an object $E \in \mathcal{C}/\vec{\mathcal{S}}$ is $\mathcal{C}/\vec{\mathcal{S}}$-injective if and only if it is $\mathcal{C}$-injective. The next few propositions are designed to show that the Grothendieck category $\mathcal{C}/\vec{\mathcal{S}}$ is locally coherent.

PROPOSITION 2.12. *An object X in $\mathcal{C}/\vec{\mathcal{S}}$ is $\mathcal{C}/\vec{\mathcal{S}}$-finitely generated if and only if it is of the form $X \cong A_{\mathcal{S}}$ for some $\mathcal{C}$-finitely generated object $A \in \mathcal{C}$. Moreover, if $X \in$ fg-$(\mathcal{C}/\vec{\mathcal{S}})$ and $Y \leqslant X$ is a $\mathcal{C}$-subobject such that $X/Y \in \vec{\mathcal{S}}$, then there is a $\mathcal{C}$-finitely generated subobject $A \leqslant Y$ such that $X/A \in \vec{\mathcal{S}}$.*

Proof. Let $A \in$ fg-$\mathcal{C}$ and suppose that $A_{\mathcal{S}} = \sum_{\mathcal{C}/\vec{\mathcal{S}}} X_i$ is a directed union in $\mathcal{C}/\vec{\mathcal{S}}$.

By the absoluteness of monomorphisms, $A_{\mathscr{S}} = \sum_{\mathscr{C}} X_i$ is a directed union in $\mathscr{C}$. For some i, $\operatorname{Im} \lambda_A \subseteq X_i$ and therefore $(\operatorname{Im} \lambda_A)_{\mathscr{S}} = A_{\mathscr{S}} \leq X_i$. That any object in fg-$(\mathscr{C}/\vec{\mathscr{S}})$ is of this form will follow from the second statement.

Let $X \in$ fg-$(\mathscr{C}/\vec{\mathscr{S}})$ and suppose that $Y \leq X$ is a $\mathscr{C}$-subobject such that $X/Y \in \vec{\mathscr{S}}$. Write $Y = \sum_{\mathscr{C}} A_i$ as a directed union of $\mathscr{C}$-finitely generated objects. Because localization preserves colimits, $X = Y_{\mathscr{S}} = \sum_{\mathscr{C}/\vec{\mathscr{S}}} (A_i)_{\mathscr{S}}$. By hypothesis, $X = (A_i)_{\mathscr{S}}$ for some i and therefore $X/A_i \in \vec{\mathscr{S}}$.

Lemma 2.13. *Let $B \in$ fg-$\mathscr{C}$. If μ: $X \leq B_{\mathscr{S}}$ is a subobject in $\mathscr{C}$ such that $\mu_{\mathscr{S}}$: $X_{\mathscr{S}} \leq B_{\mathscr{S}}$ is also $\mathscr{C}/\vec{\mathscr{S}}$-finitely generated, then there are a $\mathscr{C}$-finitely generated subobject $A \leq B$ and a commutative diagram*

$$\begin{array}{ccc} A & \rightarrowtail & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\lambda_B} \\ X & \underset{\mu}{\rightarrowtail} & B_{\mathscr{S}} \end{array}$$

such that the composition $\lambda_X \alpha$: $A \to X_{\mathscr{S}}$ is a localization of A at $\mathscr{S}$.

Proof. As $B_{\mathscr{S}}/\operatorname{Im} \lambda_B \in \vec{\mathscr{S}}$, so is $X/X \cap \operatorname{Im} \lambda_B$. By Proposition 2.12, there is a $\mathscr{C}$-finitely generated subobject $Y \leq X \cap \operatorname{Im} \lambda_B$ such that $X/Y \in \vec{\mathscr{S}}$. Let $A \leq B$ be a finitely generated subobject such that $\lambda_B(A) = Y$. Then the restriction $\alpha = \lambda_B|_A$ makes the diagram commute and the lifting α: $A \to X_{\mathscr{S}}$ is a localization of A at $\mathscr{S}$.

The lemma implies that if B is $\mathscr{C}$-finitely generated and η: $X \to B_{\mathscr{S}}$ is a $\mathscr{C}/\vec{\mathscr{S}}$-epimorphism with X a $\mathscr{C}/\vec{\mathscr{S}}$-finitely generated object, then by replacing B with an appropriate finitely generated subobject, we may assume that $\operatorname{Im}_{\mathscr{C}} \lambda_B \leq \operatorname{Im}_{\mathscr{C}} \eta$. Of course, if B is coherent, then so is the finitely generated subobject. The lemma also yields the following.

Proposition 2.14. *Let $B \in$ fg-$\mathscr{C}$ and let $0 \to X \to B_{\mathscr{S}} \to Z \to 0$ be a short exact sequence in $\mathscr{C}/\vec{\mathscr{S}}$ with X a $\mathscr{C}/\vec{\mathscr{S}}$-finitely generated object. Then there is a short exact sequence in $\mathscr{C}$ of $\mathscr{C}$-finitely generated objects $0 \to A \to B \to C \to 0$ of which the above sequence is a localization.*

It is clear that if the object B above is assumed to be coherent, then the short exact sequence $0 \to A \to B \to C \to 0$ lies in coh-$\mathscr{C}$. The following result seems to be new and has been independently noted by Krause [**22**].

Proposition 2.15. *If C is $\mathscr{C}$-coherent, then $C_{\mathscr{S}}$ is $\mathscr{C}/\vec{\mathscr{S}}$-coherent.*

Proof. We shall prove that if C is $\mathscr{C}$-coherent, then $C_{\mathscr{S}}$ is $\mathscr{C}/\vec{\mathscr{S}}$-finitely presented. It then follows from Lemma 2.13 that $C_{\mathscr{S}}$ is $\mathscr{C}/\vec{\mathscr{S}}$-coherent because every $\mathscr{C}/\vec{\mathscr{S}}$-finitely generated subobject has the form $A_{\mathscr{S}}$ for some $\mathscr{C}$-finitely generated subobject $A \leq C$. Then A is $\mathscr{C}$-coherent and $A_{\mathscr{S}}$ is $\mathscr{C}/\vec{\mathscr{S}}$-finitely presented.

Suppose η: $X \to C_{\mathscr{S}}$ is a $\mathscr{C}/\vec{\mathscr{S}}$-epimorphism and X a $\mathscr{C}/\vec{\mathscr{S}}$-finitely generated object. By the remarks following Lemma 2.13, we may assume that $\operatorname{Im}_{\mathscr{C}} \eta \leq \operatorname{Im}_{\mathscr{C}} \lambda_C$. To prove that $C_{\mathscr{S}}$ is $\mathscr{C}/\vec{\mathscr{S}}$-finitely presented, it needs to be shown that

$Y = \operatorname{Ker} \eta$ is $\mathscr{C}/\vec{\mathscr{S}}$-finitely generated. Write $Y = \sum_{\mathscr{C}} Y_i$ as a directed union of $\mathscr{C}$-finitely generated subobjects. By Proposition 1.2, the morphism λ_C: $C \to \operatorname{Im} \eta \cong X/Y \cong X/\sum_{\mathscr{C}} Y_i \cong \varinjlim X/Y_i$ factors through one of the X/Y_i. Because the localization

$$\lambda_C\colon\ C \longrightarrow X/Y_i \xrightarrow{\ \alpha\ } C_{\mathscr{S}}$$

becomes a $\mathscr{C}/\vec{\mathscr{S}}$-isomorphism upon localization, $\alpha_{\mathscr{S}}$ is a $\mathscr{C}/\vec{\mathscr{S}}$-split-epimorphism. As $(X/Y_i)_{\mathscr{S}}$ is $\mathscr{C}/\vec{\mathscr{S}}$-finitely generated, so is $\operatorname{Ker}_{\mathscr{C}/\vec{\mathscr{S}}} \alpha_{\mathscr{S}}$. Localizing the factorization

$$\eta\colon\ X \xrightarrow{\ \pi\ } X/Y_i \xrightarrow{\ \alpha\ } C_{\mathscr{S}}$$

gives that $\eta = \eta_{\mathscr{S}} = (\alpha\pi)_{\mathscr{S}} = \alpha_{\mathscr{S}}\pi_{\mathscr{S}}$ and because $\operatorname{Ker}_{\mathscr{C}/\vec{\mathscr{S}}} \pi_{\mathscr{S}} = (Y_i)_{\mathscr{S}}$ is also $\mathscr{C}/\vec{\mathscr{S}}$-finitely generated, so is $\operatorname{Ker} \eta$.

If $X \in \mathscr{C}$ is a direct limit $X \cong \varinjlim C_i$ of coherent objects C_i, then localization gives a representation of $X_{\mathscr{S}} \cong \varinjlim_{\mathscr{C}/\vec{\mathscr{S}}} (C_i)_{\mathscr{S}}$ as a direct limit of $\mathscr{C}/\vec{\mathscr{S}}$-coherent objects. In particular, every $\mathscr{C}/\vec{\mathscr{S}}$-coherent object X is a $\mathscr{C}/\vec{\mathscr{S}}$-quotient of some $C_{\mathscr{S}}$ with C a $\mathscr{C}$-coherent object. By Proposition 2.14, X also has this form. These observations give the following result.

THEOREM 2.16. *The category $\mathscr{C}/\vec{\mathscr{S}}$ of $\vec{\mathscr{S}}$-closed objects of $\mathscr{C}$ is a locally coherent Grothendieck category. An object of $\mathscr{C}/\vec{\mathscr{S}}$ is coherent if and only if it has the form $C_{\mathscr{S}}$ for some $\mathscr{C}$-coherent object C.*

For later reference, we prove next that the category $\mathscr{C}/\vec{\mathscr{S}}$ as it is defined here coincides with the standard definition (cf. [**12**]). Let A and B be coherent objects of $\mathscr{C}$. If $A' \leqslant A$ is a coherent subobject such that $A/A' \in \mathscr{S}$, then it is clear that the localization of A' at $\mathscr{S}$ factors though A as λ_A: $A' \leqslant A \to A_{\mathscr{S}}$. Dually, if $B' \leqslant B$ is a coherent subobject in $\mathscr{S}$, then the localization of B at $\mathscr{S}$ factors through the localization of B/B' as λ_B: $B \to B/B' \to (B/B')_{\mathscr{S}}$. Thus $A_{\mathscr{S}} \cong (A')_{\mathscr{S}}$ and $B_{\mathscr{S}} \cong (B/B')_{\mathscr{S}}$ and localization at $\mathscr{S}$ gives the following morphism of abelian groups:

$$(-)_{\mathscr{S}}\colon\ \operatorname{Hom}_{\mathscr{C}}(A', B/B') \to \operatorname{Hom}_{\mathscr{C}/\vec{\mathscr{S}}}(A_{\mathscr{S}}, B_{\mathscr{S}}). \tag{1}$$

Define a partial order

$$(A', B/B') \sqsubseteq (A'', B/B'')$$

on the set of pairs $(A', B/B')$ as above, by $A'' \leqslant A'$ and $B'' \geqslant B'$. This partial order is directed because two pairs $(A', B/B')$ and $(A'', B/B'')$ have the common upper bound $(A' \cap A'', B/(B' + B''))$. When this relation holds, a morphism of abelian groups

$$\operatorname{Hom}_{\mathscr{C}}(A', B/B') \to \operatorname{Hom}_{\mathscr{C}}(A'', B/B'')$$

is induced by restriction to A'' and congruence modulo B''/B'. These are the structural morphisms of a directed system indexed by the pairs $(A', B/B')$.

PROPOSITION 2.17. *Let A and B be coherent objects of $\mathscr{C}$. The morphism of abelian groups*

$$(-)_{\mathscr{S}}\colon\ \varinjlim \operatorname{Hom}_{\mathscr{C}}(A', B/B') \to \operatorname{Hom}_{\mathscr{C}/\vec{\mathscr{S}}}(A_{\mathscr{S}}, B_{\mathscr{S}})$$

induced by the compatible family (1) *of morphisms is an isomorphism.*

Proof. With intent to prove the morphism monic, assume that $\alpha \in \varinjlim \operatorname{Hom}_{\mathcal{C}}(A', B/B')$ is such that $\alpha_{\mathcal{S}} = 0$. This means that for some representative $\alpha' \in \operatorname{Hom}_{\mathcal{C}}(A', B/B')$ of α, we have that $\alpha'_{\mathcal{S}} = 0$. Thus $\operatorname{Im} \alpha'_{\mathcal{S}} \in \vec{\mathcal{S}}$. Now $\operatorname{Im} \alpha'_{\mathcal{S}} \leqslant B/B'$ is coherent and so there is a coherent subobject $B'' \leqslant B$ such that $B' \leqslant B''$ and $B''/B' = \operatorname{Im} \alpha'_{\mathcal{S}}$. Then the morphism induced by α' in $\operatorname{Hom}_{\mathcal{C}}(A', B/B'')$ is zero and hence $\alpha = 0$.

With intent to verify surjectivity, assume $\eta \in \operatorname{Hom}_{\mathcal{C}/\vec{\mathcal{S}}}(A_{\mathcal{S}}, B_{\mathcal{S}})$. If $\lambda_B\colon B \to B_{\mathcal{S}}$ is the localization of B at $\mathcal{S}$, then the subobject $Y = \operatorname{Im}_{\mathcal{C}} \lambda_B \cong B/t(B)$ of $B_{\mathcal{S}}$ has the property that $B_{\mathcal{S}}/Y \in \vec{\mathcal{S}}$. We shall find a coherent subobject $A' \leqslant A$ such that $A/A' \in \mathcal{S}$ and the image of the restriction

$$A' \leqslant A \xrightarrow{\lambda_A} A_{\mathcal{S}} \xrightarrow{\eta} B_{\mathcal{S}}$$

is a subobject of $Y \cong B/t(B)$. By Proposition 1.2, this morphism factors through B/B' for some $B' \in \mathcal{S}$ and the commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{\alpha} & B/B' \\ {\scriptstyle \lambda_{A'}}\downarrow & & \downarrow{\scriptstyle \lambda_{(B/B')}} \\ A_{\mathcal{S}} & \xrightarrow[\eta]{} & B_{\mathcal{S}} \end{array}$$

gives a preimage of η, $\alpha_{\mathcal{S}} = \eta$. But such a subobject $A' \leqslant A$ is given by Lemma 2.13 with $X = \eta^{-1}(Y)$. Clearly $A_{\mathcal{S}}/X \in \vec{\mathcal{S}}$ and so $(A')_{\mathcal{S}} = X_{\mathcal{S}} = A_{\mathcal{S}}$ and hence $A/A' \in \text{coh-}\vec{\mathcal{S}} = \mathcal{S}$.

3. *The Ziegler spectrum*

In this section, a topological space $\operatorname{Zg}(\mathcal{C})$ called the Ziegler spectrum is associated to the locally coherent Grothendieck category $\mathcal{C}$. This space was introduced by Ziegler in his model-theoretic analysis [**37**] of modules.

Let $\mathcal{S}_i \subseteq \text{coh-}\mathcal{C}$, for $i \in I$, be a collection of Serre subcategories of coh-$\mathcal{C}$. Evidently, the intersection $\bigcap_{i \in I} \mathcal{S}_i$ is also a Serre subcategory of coh-$\mathcal{C}$. So if $\mathcal{X} \subseteq \text{coh-}\mathcal{C}$ is an arbitrary subcategory, we may talk about the smallest Serre subcategory of coh-$\mathcal{C}$ to contain $\mathcal{X}$. This Serre subcategory is denoted by

$$\sqrt{\mathcal{X}} = \bigcap \{\mathcal{S} \subseteq \text{coh-}\mathcal{C}\colon\ \mathcal{S} \supseteq \mathcal{X} \text{ is Serre}\}.$$

To describe $\sqrt{\mathcal{X}}$ intrinsically, we need the notion of a subquotient.

Definition [**37**, Definition, p. 156]. Given objects $A, B \in \text{coh-}\mathcal{C}$, we say that A is a *subquotient* of B, or $A \prec B$, if there is a filtration of B by coherent subobjects

$$B = B_0 \geqslant B_1 \geqslant B_2 \geqslant 0$$

such that $A \cong B_1/B_2$. In other words, A is isomorphic to a coherent subobject of a coherent quotient object of B or, equivalently, A is isomorphic to a coherent quotient object of a coherent subobject of B. We shall use the notion of subquotient only in the category of coherent objects.

It is easy to see that the relation $A \prec B$ is transitive and it is immediate from the definition of a Serre subcategory $\mathcal{S}$ that if $B \in \mathcal{S}$ and A is a subquotient of B, then $A \in \mathcal{S}$. In particular, if $B \in \mathcal{X} \subseteq \text{coh-}\mathcal{C}$, then $A \in \sqrt{\mathcal{X}}$.

PROPOSITION 3.1. *A coherent object $C \in \sqrt{\mathcal{X}}$ if and only if there are a finite filtration of C by coherent subobjects*

$$C = C_0 \geqslant C_1 \geqslant \dots \geqslant C_n = 0$$

and, for every $i < n$, $A_i \in \mathcal{X}$ such that $C_i / C_{i+1} \prec A_i$.

Proof. Suppose that C has such a filtration. Since each subquotient C_i/C_{i+1} is in $\sqrt{\mathcal{X}}$ and $\sqrt{\mathcal{X}}$ is closed under extensions, $C \in \sqrt{\mathcal{X}}$. Conversely, it is easy to check that the class of those C with such a filtration satisfies the axioms for a Serre subcategory containing $\mathcal{X}$.

Denote by $\text{Zg}(\mathcal{C})$ the set of indecomposable injective objects of $\mathcal{C}$ (up to isomorphism). This is indeed a set because every $E \in \text{Zg}(\mathcal{C})$ is the injective envelope of a finitely generated object in $\mathcal{C}$ and fg-$\mathcal{C}$ is skeletally small. To an arbitrary subcategory $\mathcal{X} \subseteq \text{coh-}\mathcal{C}$, we associate the subset of $\text{Zg}(\mathcal{C})$,

$$\mathcal{O}(\mathcal{X}) = \{E \in \text{Zg}(\mathcal{C})\colon \text{ for some } C \in \mathcal{X}, \text{Hom}_{\mathcal{C}}(C, E) \neq 0\}.$$

If $\mathcal{X} = \{C\}$ is a singleton, we abbreviate $\mathcal{O}(\mathcal{X})$ to $\mathcal{O}(C)$. Thus $\mathcal{O}(\mathcal{X}) = \bigcup_{C \in \mathcal{X}} \mathcal{O}(C)$.

PROPOSITION 3.2. *If A and B are coherent objects and A is a subquotient of B, then $\mathcal{O}(A) \subseteq \mathcal{O}(B)$. If $0 \to A_1 \to B \to A_2 \to 0$ is a short exact sequence in* coh-$\mathcal{C}$, *then $\mathcal{O}(B) = \mathcal{O}(A_1) \cup \mathcal{O}(A_2)$.*

Proof. Let C be a coherent quotient object of B and let $A \leqslant C$ be coherent. If $E \in \mathcal{O}(A)$, then a non-zero morphism $\eta\colon A \to E$ extends to C. This yields a non-zero morphism $\eta'\colon B \to E$. To prove the second statement, apply the fact that for $E \in \text{Zg}(\mathcal{C})$, the functor $(-, E)$ is exact on $\mathcal{C}$.

The next result follows from Propositions 3.1 and 3.2.

PROPOSITION 3.3. *For any subcategory $\mathcal{X} \subseteq$* coh-$\mathcal{C}$, $\mathcal{O}(\mathcal{X}) = \mathcal{O}(\sqrt{\mathcal{X}})$.

Proof. Let $C \in \sqrt{\mathcal{X}}$ and consider a filtration of C as given by Proposition 3.1.

By Proposition 3.2, we have that $\mathcal{O}(C) = \bigcup_{i<n} \mathcal{O}(C_i/C_{i+1}) \subseteq \bigcup_{i<n} \mathcal{O}(A_i)$ with $A_i \in \mathcal{X}$. Thus $\mathcal{O}(\sqrt{\mathcal{X}}) = \bigcup_{C \in \sqrt{\mathcal{X}}} \mathcal{O}(C) \subseteq \mathcal{O}(\mathcal{X})$.

Consequently, there is no loss in generality if we restrict the discussion to subsets of the form $\mathcal{O}(\mathcal{S})$ where $\mathcal{S} \subseteq \text{coh-}\mathcal{C}$ is a Serre subcategory. In that case,

$$\mathcal{O}(\mathcal{S}) = \{E \in \text{Zg}(\mathcal{C}): \ t_{\mathcal{S}}(E) \neq 0\}.$$

THEOREM 3.4 (Ziegler [**37**, Theorem 4.9]). *The collection of subsets of* Zg($\mathcal{C}$),

$$\{\mathcal{O}(\mathcal{S}): \ \mathcal{S} \subseteq \text{coh-}\mathcal{C} \textit{ is a Serre subcategory}\},$$

satisfies the axioms for the open sets of a topology on Zg($\mathcal{C}$). *This topological space is called the* Ziegler spectrum *of* $\mathcal{C}$.

Proof. First note that $\mathcal{O}(0) = \varnothing$ and $\mathcal{O}(\text{coh-}\mathcal{C}) = \text{Zg}(\mathcal{C})$. By Proposition 3.3, $\bigcup_{i \in I} \mathcal{O}(\mathcal{S}_i) = \mathcal{O}(\bigcup_{i \in I} \mathcal{S}_i) = \mathcal{O}(\sqrt{\bigcup_{i \in I} \mathcal{S}_i})$. It remains to be shown that $\mathcal{O}(\mathcal{S}_1) \cap \mathcal{O}(\mathcal{S}_2) = \mathcal{O}(\mathcal{S}_1 \cap \mathcal{S}_2)$. Suppose that $E \in \mathcal{O}(\mathcal{S}_1) \cap \mathcal{O}(\mathcal{S}_2)$. As E is uniform, $t_{\mathcal{S}_1}(E) \cap t_{\mathcal{S}_2}(E) \neq 0$, so consider a finitely generated non-zero $X \leqslant t_{\mathcal{S}_1}(E) \cap t_{\mathcal{S}_2}(E)$. There is an epimorphism η_1: $S_1 \to X$ with $S_1 \in \mathcal{S}_1$. By Proposition 2.3, this morphism η factors through a quotient $S_2 \in \mathcal{S}_2$ of S_1. But then $S_2 \in \mathcal{S}_1 \cap \mathcal{S}_2$ and therefore $E \in \mathcal{O}(\mathcal{S}_1 \cap \mathcal{S}_2)$.

COROLLARY 3.5 (Ziegler [**37**, Theorem 4.9]). *The collection of open subsets*

$$\{\mathcal{O}(C): \ C \in \text{coh-}\mathcal{C}\}$$

satisfies the axioms for a basis of open subsets of the Ziegler spectrum. Furthermore, $\mathcal{O}(C) = \varnothing$ *if and only if* $C = 0$.

Proof. The first statement is a consequence of the fact that every open subset $\mathcal{O}(\mathcal{X}) = \bigcup_{C \in \mathcal{X}} \mathcal{O}(C)$ is a union of open sets from this collection. The second derives from the observation that every non-zero $C \in \text{coh-}\mathcal{C}$ has a simple quotient object S whose injective envelope $E(S) \in \mathcal{O}(C)$.

The maximal Ziegler spectrum of $\mathcal{C}$, denoted by max($\mathcal{C}$), is the subset of Zg($\mathcal{C}$) consisting of those indecomposable injectives that are injective envelopes of simple objects. The proof of Corollary 3.5 indicates that max($\mathcal{C}$) is a dense subset of Zg($\mathcal{C}$). For the next result, we shall need the notation $I(\mathcal{S}) = \text{Zg}(\mathcal{C}) \setminus \mathcal{O}(\mathcal{S})$ for the closed set which is the complement of the open set $\mathcal{O}(\mathcal{S})$.

PROPOSITION 3.6. *Let* $\mathcal{S} \subseteq \text{coh-}\mathcal{C}$ *be a Serre subcategory. The inclusion functor* $\mathcal{C}/\vec{\mathcal{S}} \subseteq \mathcal{C}$ *induces a homeomorphism* h: $\text{Zg}(\mathcal{C}/\vec{\mathcal{S}}) \to I(\mathcal{S})$ *from the Ziegler spectrum of* $\mathcal{C}/\vec{\mathcal{S}}$ *onto the closed set* $I(\mathcal{S})$ *endowed with the relative subspace topology. Furthermore,* $h[\mathcal{O}(C_{\mathcal{S}})] = I(\mathcal{S}) \cap \mathcal{O}(C)$ *for* $C \in \text{coh-}\mathcal{C}$.

Proof. By the comments following Proposition 2.11, $E \in \text{Zg}(\mathcal{C}/\vec{\mathcal{S}})$ if and only if $E \in \text{Zg}(\mathcal{C})$ and E is $\vec{\mathcal{S}}$-torsion-free, that is, $t_{\mathcal{S}}(E) = 0$, that is, $E \in I(\mathcal{S})$. Thus h is a bijection. That it is a homeomorphism follows from the second statement which is a consequence of the left adjoint property of the localization functor. If $E \in \text{Zg}(\mathcal{C}/\vec{\mathcal{S}})$ and $C \in \text{coh-}\mathcal{C}$, then $\text{Hom}_{\mathcal{C}}(C, E) \cong \text{Hom}_{\mathcal{C}/\vec{\mathcal{S}}}(C_{\mathcal{S}}, E)$.

Given an open set $\mathcal{O} \subseteq \mathrm{Zg}(\mathcal{C})$, consider the subcategory

$$\mathcal{S}_{\mathcal{O}} = \{C \in \text{coh-}\mathcal{C}\colon\ \mathcal{O}(C) \subseteq \mathcal{O}\}$$

of coh-$\mathcal{C}$. By Proposition 3.2, it is a Serre subcategory.

LEMMA 3.7. *For every Serre subcategory $\mathcal{S} \subseteq$ coh-$\mathcal{C}$, $\mathcal{S} = \mathcal{S}_{\mathcal{O}(\mathcal{S})}$.*

Proof. We have

$$\begin{aligned} C \in \mathcal{S}_{\mathcal{O}(\mathcal{S})} \quad &\Leftrightarrow \quad \mathcal{O}(C) \cap I(\mathcal{S}) = \varnothing \\ &\Leftrightarrow \quad \mathcal{O}(C_{\mathcal{S}}) = \varnothing \quad (\text{in } \mathrm{Zg}(\mathcal{C}/\vec{\mathcal{S}})) \\ &\Leftrightarrow \quad C_{\mathcal{S}} = 0 \quad (\text{by Proposition 3.6}) \\ &\Leftrightarrow \quad C \in \mathcal{S}. \end{aligned}$$

THEOREM 3.8. *There is an inclusion-preserving bijective correspondence between the Serre subcategories $\mathcal{S}$ of* coh-$\mathcal{C}$ *and the open subsets $\mathcal{O}$ of* $\mathrm{Zg}(\mathcal{C})$. *This correspondence is given by the functions*

$$\mathcal{S} \mapsto \mathcal{O}(\mathcal{S}), \quad \mathcal{O} \mapsto \mathcal{S}_{\mathcal{O}},$$

which are mutual inverses.

Proof. The previous lemma shows that $\mathcal{O} \mapsto \mathcal{S}_{\mathcal{O}}$ is the left inverse of $\mathcal{S} \mapsto \mathcal{O}(\mathcal{S})$. It is also a right inverse because

$$\mathcal{O}(\mathcal{S}_{\mathcal{O}}) = \bigcup \{\mathcal{O}(C)\colon\ C \in \mathcal{S}_{\mathcal{O}}\} = \bigcup \{\mathcal{O}(C)\colon\ \mathcal{O}(C) \subseteq \mathcal{O}\} = \mathcal{O}.$$

The first and last equalities hold because the $\mathcal{O}(C)$ constitute a basis.

COROLLARY 3.9 (Ziegler [**37**, Theorem 4.9]). *An open subset $\mathcal{O}$ of* $\mathrm{Zg}(\mathcal{C})$ *is quasi-compact if and only if it is one of the basic open subsets $\mathcal{O}(C)$ with $C \in$* coh-$\mathcal{C}$.

Proof. If $\mathcal{O} = \bigcup_{i \in I} \mathcal{O}(C_i)$ is quasi-compact, then for some finite subset $J \subseteq I$, $\mathcal{O} = \bigcup_{i \in J} \mathcal{O}(C_i) = \mathcal{O}(\coprod_{i \in J} C_i)$. Conversely, if $\mathcal{O}(C) = \bigcup_{i \in I} \mathcal{O}(C_i) = \mathcal{O}(\{C_i\colon\ i \in I\})$, then by Theorem 3.8 and Proposition 3.3, $C \in \bigvee\{C_i\colon\ i \in I\}$. By Proposition 3.1, there is a finite filtration of C by coherent subobjects such that each factor of the filtration is a subquotient of some C_i. Since only finitely many of the C_i are needed, there is a finite subset J of I such that $C \in \bigvee\{C_i\colon\ i \in J\}$ and therefore $\mathcal{O}(C) = \bigcup_{i \in J} \mathcal{O}(C_i)$.

Let $M \in \mathcal{C}$ be coh-injective and denote by $I(M)$ the closed set $I(\mathcal{S}(M))$ where $\mathcal{S}(M)$ is the Serre subcategory $\mathcal{S}(M) = \{C \in \text{coh-}\mathcal{C}\colon\ (C, M) = 0\}$. The set $I(M)$ is called the closed set of M. By Theorem 3.8, $I(M)$ may be characterized as the closed subset of $\mathrm{Zg}(\mathcal{C})$ satisfying the condition that for each $C \in$ coh-$\mathcal{C}$, $\mathcal{O}(C) \cap I(M) = \varnothing$ if and only if $(C, M) = 0$. If E is an injective indecomposable object of $\mathcal{C}$, that is, if $E \in \mathrm{Zg}\,\mathcal{C}$, then $I(E)$ is simply the topological closure of the point $\{E\}$.

PROPOSITION 3.10. *Let $\mathcal{S} \subseteq$ coh-$\mathcal{C}$ be a Serre subcategory and $M \in \mathcal{C}$ a coh-injec-*

tive $\vec{\mathcal{S}}$-torsion-free object. Then M is $\vec{\mathcal{S}}$-closed and it is a coh-injective object of $\mathcal{C}/\vec{\mathcal{S}}$. If h: $\mathrm{Zg}(\mathcal{C}/\vec{\mathcal{S}}) \to \mathrm{Zg}(\mathcal{C})$ is the continuous map of Proposition 3.6, *then* $h[I_{\mathcal{C}/\vec{\mathcal{S}}}(M)] = I_{\mathcal{C}}(M)$.

Proof. As M is an $\vec{\mathcal{S}}$-torsion-free object, we have that $M \leq M_{\mathcal{S}}$ and $X = M_{\mathcal{S}}/M \in \vec{\mathcal{S}}$. But if $S \in \mathcal{S}$, then the long exact sequence

$$0 \to (S, M) = 0 \to (S, M_{\mathcal{S}}) = 0 \to (S, X) \to \mathrm{Ext}^1(S, M) = 0$$

proves that $(S, X) = 0$. By Proposition 2.4, $X = 0$ and $M = M_{\mathcal{S}} \in \mathcal{C}/\vec{\mathcal{S}}$.

To verify that M is $(\mathcal{C}/\vec{\mathcal{S}})$-coh-injective, we use Theorem 2.16 and Proposition 2.14 to note that every short exact sequence in coh-$(\mathcal{C}/\vec{\mathcal{S}})$ is the localization of a short exact sequence in coh-$(\mathcal{C})$,

$$0 \to A \to B \to C \to 0.$$

As M is $\mathcal{C}$-coh-injective, the functor $(-, M)$ is exact on coh-$\mathcal{C}$. Thus the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(C, M) \to \mathrm{Hom}_{\mathcal{C}}(B, M) \to \mathrm{Hom}_{\mathcal{C}}(A, M) \to 0$$

is exact. But this sequence is isomorphic to the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}/\vec{\mathcal{S}}}(C_{\mathcal{S}}, M) \to \mathrm{Hom}_{\mathcal{C}/\vec{\mathcal{S}}}(B_{\mathcal{S}}, M) \to \mathrm{Hom}_{\mathcal{C}/\vec{\mathcal{S}}}(A_{\mathcal{S}}, M) \to 0.$$

To show that $h[I_{\mathcal{C}/\vec{\mathcal{S}}}(M)] = I(M)$, note that

$$\begin{aligned} \mathcal{O}(C) \cap h[I_{\mathcal{C}/\vec{\mathcal{S}}}(M)] \neq \varnothing \quad &\Leftrightarrow \quad \mathcal{O}(C_{\mathcal{S}}) \cap h[I_{\mathcal{C}/\vec{\mathcal{S}}}(M)] \neq \varnothing \quad \text{(by Proposition 3.6)} \\ &\Leftrightarrow \quad \mathrm{Hom}_{\mathcal{C}/\vec{\mathcal{S}}}(C_{\mathcal{S}}, M) \neq 0 \\ &\Leftrightarrow \quad \mathrm{Hom}_{\mathcal{C}}(C, M) \neq 0. \end{aligned}$$

By the characterization above of the closed set $I(M)$, we get that $h[I_{\mathcal{C}/\vec{\mathcal{S}}}(M)] = I(M)$.

There is always a coh-injective object $M \in \mathcal{C}$ such that $\mathrm{Zg}(\mathcal{C}) = I(M)$. One may take any coproduct of injective indecomposable objects

$$M = \coprod_{E \in \mathcal{D}} E$$

indexed by a dense subset $\mathcal{D}$ (for example, $\max(\mathcal{C})$) of the Ziegler spectrum. If $\mathcal{S} \subseteq$ coh-$\mathcal{C}$ is a Serre subcategory, then applying this argument to $\mathcal{C}/\vec{\mathcal{S}}$ and the homeomorphism h: $\mathrm{Zg}(\mathcal{C}/\vec{\mathcal{S}}) \to I(\mathcal{S})$ gives the following.

Corollary 3.11 (Ziegler [**37**, Corollary 4.10]). *Every closed set $I \subseteq \mathrm{Zg}(\mathcal{C})$ is the closed set $I(M)$ of some coh-injective object $M \in \mathcal{C}$. Thus every Serre subcategory $\mathcal{S} \subseteq$ coh-$\mathcal{C}$ has the form $\mathcal{S}(M)$ for some coh-injective object $M \in \mathcal{C}$.*

For $C \in$ coh-$\mathcal{C}$, define

$$\mathrm{Supp}(C) := \{M\colon\ M \text{ is coh-injective, } (C, M) \neq 0\}.$$

Equivalently,

$$M \in \mathrm{Supp}(C) \quad \Leftrightarrow \quad \mathcal{O}(C) \cap I(M) = \mathcal{O}(C_{\mathcal{S}(M)}) \neq \varnothing \quad \Leftrightarrow \quad C \notin \mathcal{S}(M).$$

If $M \in \mathcal{C}$ is a coh-injective object, define $\mathcal{C}(M)$ to be the quotient category $\mathcal{C}/\vec{\mathcal{S}}(M)$ of $\mathcal{C}$ by $\vec{\mathcal{S}}(M)$.

COROLLARY 3.12. *The following are equivalent for two coherent objects A and B of* $\mathscr{C}$*:*

(1) $\mathcal{O}(A) \subseteq \mathcal{O}(B)$*;*

(2) $A \in \sqrt{B}$ $(:= \sqrt{\{B\}})$*;*

(3) *there is a finite filtration of A by coherent subobjects*

$$A = A_0 \geqslant A_1 \geqslant \dots \geqslant A_n = 0$$

such that each of the factors A_i/A_{i+1} *is a subquotient of B;*

(4) $\mathrm{Supp}(A) \subseteq \mathrm{Supp}(B)$*;*

(5) *for every coh-injective* $M \in \mathscr{C}$, $\mathcal{O}(A_{\mathscr{S}(M)}) \subseteq \mathcal{O}(B_{\mathscr{S}(M)})$ *in* $\mathscr{C}(M)$*;*

(6) *for every Serre subcategory* $\mathscr{S} \subseteq \text{coh-}\mathscr{C}$, $\mathcal{O}(A_{\mathscr{S}}) \subseteq \mathcal{O}(B_{\mathscr{S}})$ *in* $\mathrm{Zg}(\mathscr{C}/\vec{\mathscr{S}})$*;*

(7) *for every* $\cap$*-irreducible Serre subcategory* $\mathscr{S} \subseteq \text{coh-}\mathscr{C}$, $\mathcal{O}(A_{\mathscr{S}}) \subseteq \mathcal{O}(B_{\mathscr{S}})$ *in* $\mathrm{Zg}(\mathscr{C}/\vec{\mathscr{S}})$*;*

(8) *for every injective indecomposable* $E \in \mathrm{Zg}(\mathscr{C})$, $\mathcal{O}(A_{\mathscr{S}(E)}) \subseteq \mathcal{O}(B_{\mathscr{S}(E)})$ *in* $\mathrm{Zg}(\mathscr{C}(E))$.

Proof. (1)$\Leftrightarrow$(2). We have $\mathcal{O}(A) \subseteq \mathcal{O}(B)$ if and only if $A \in \mathscr{S}_{\mathcal{O}(B)} = \sqrt{B}$.

(2)$\Leftrightarrow$(3). This is a special case of Proposition 3.1.

(3)$\Leftrightarrow$(4). Because the coh-injective objects of $\mathscr{C}$ behave like injective objects with respect to the coherent objects, one can imitate the proof of Proposition 3.2 to show that $\mathrm{Supp}(A) = \bigcup_{i<n} \mathrm{Supp}(A_i/A_{i+1})$ and that for each *i*, $\mathrm{Supp}(A_i/A_{i+1}) \subseteq \mathrm{Supp}(B)$.

(4)$\Rightarrow$(5). If $\mathrm{Supp}(A) \subseteq \mathrm{Supp}(B)$, then $\mathcal{O}(A) \subseteq \mathcal{O}(B)$ and therefore $\mathcal{O}(A_{\mathscr{S}(M)}) = \mathcal{O}(A) \cap I(M) \subseteq \mathcal{O}(B) \cap I(M) = \mathcal{O}(B_{\mathscr{S}(M)})$.

(5)$\Rightarrow$(6)$\Rightarrow$(7). These are trivial.

(7)$\Rightarrow$(8). If $E \in \mathrm{Zg}(\mathscr{C})$, then $I(E)$, being the closure of a point, is not the union of two proper closed subsets. By Theorem 3.8, the Serre subcategory $\mathscr{S}(E) \subseteq \text{coh-}\mathscr{C}$ is $\cap$-irreducible, that is, it is not the intersection of two properly larger Serre subcategories.

(8)$\Rightarrow$(1). If $E = \mathcal{O}(A)$, then $\mathcal{O}(A_{\mathscr{S}(E)}) \neq \varnothing$ and so by hypothesis $\mathcal{O}(B_{\mathscr{S}(E)}) \neq \varnothing$. But then $E \in \mathcal{O}(B)$.

We shall give an example later showing that the last condition cannot be strengthened to $E \in \max(\mathscr{C})$.

COROLLARY 3.13 (Ziegler [**37**, Theorem 4.9]). *Let* $E \in \mathrm{Zg}(\mathscr{C})$ *and* $C \in \text{coh-}\mathscr{C}$ *be such that* $E \in \mathcal{O}(C)$. *A local system of open neighbourhoods of E is given by the collection*

$$\{\mathcal{O}(A)\colon\ E \in \mathcal{O}(A),\, A \prec C\}.$$

Proof. Let $\mathcal{O} \in \mathrm{Zg}(\mathscr{C})$ be open with $E \in \mathcal{O}$. Choose $A \in \text{coh-}\mathscr{C}$ such that $E \in \mathcal{O}(A) \subseteq \mathcal{O} \cap \mathcal{O}(C)$. As in Proposition 3.12.4, there is a filtration of A by coherent subobjects

$$A = A_0 \geqslant A_1 \geqslant \dots \geqslant A_n = 0$$

such that each A_i/A_{i+1} is a subquotient of C. By Proposition 3.2, there is an $i < n$ such that $E \in \mathcal{O}(A_i/A_{i+1})$. But then $E \in \mathcal{O}(A_i/A_{i+1}) \subseteq \mathcal{O}(A) \cap \mathcal{O}(C)$.

The injective indecomposable objects $E \in \mathrm{Zg}(\mathscr{C})$ are the maximal uniform objects of $\mathscr{C}$ in the sense that they possess no proper uniform extensions. Therefore

$$\mathrm{Zg}(\vec{\mathscr{S}}) = \{t_{\mathscr{S}}(E)\colon\ E \in \mathrm{Zg}(\mathscr{C})\}$$

and there exists a bijection $E_{\mathscr{C}}\colon\ \mathrm{Zg}(\vec{\mathscr{S}}) \to \mathscr{O}(\mathscr{S})$ defined by sending the maximal uniform object $t_{\mathscr{S}}(E)$ of $\vec{\mathscr{S}}$ to its $\mathscr{C}$-injective hull $E_{\mathscr{C}}(t_{\mathscr{S}}(E)) = E$. Because the torsion functor $t_{\mathscr{S}}$ is the right adjoint of the inclusion functor $\vec{\mathscr{S}} \subseteq \mathscr{C}$, we have that for $C \in \mathscr{S} = \text{coh-}\vec{\mathscr{S}}$ and $E \in \mathrm{Zg}(\mathscr{C})$, there is an isomorphism

$$\mathrm{Hom}_{\vec{\mathscr{S}}}(C, t_{\mathscr{S}}(E)) \cong \mathrm{Hom}_{\mathscr{C}}(C, E),$$

which proves that $E_{\mathscr{C}}[\mathscr{O}(C)] = \mathscr{O}(C) \subseteq \mathrm{Zg}(\mathscr{C})$ and hence that $E_{\mathscr{C}}\colon\ \mathrm{Zg}(\vec{\mathscr{S}}) \to \mathscr{O}(\mathscr{S})$ is a homeomorphism.

4. *The Ziegler spectrum of a ring*

Let R be a ring. The left Ziegler spectrum of R is the topological space $\mathrm{Zg}({}_R\mathscr{C})$. In this section, we describe some points of the Ziegler spectrum of certain rings. The points of $\mathrm{Zg}({}_R\mathscr{C})$ are the injective indecomposable objects E of ${}_R\mathscr{C} =$ (mod-R, Ab). If ${}_RM \in R$-Mod, let us abbreviate the closed set $I(-\otimes_R M)$ to $I({}_RM)$ and the related Serre subcategory $\mathscr{S}(-\otimes_R M)$ of coh-$({}_R\mathscr{C})$ to $\mathscr{S}({}_RM)$.

Example. If $R = R_1 \times R_2$ is the Cartesian product of the rings R_1 and R_2, then the left Ziegler spectrum of R is a disjoint union of open sets

$$\mathrm{Zg}({}_R\mathscr{C}) = \mathscr{O}({}_RR_1, -) \,\dot\cup\, \mathscr{O}({}_RR_2, -)$$

which are homeomorphic to $\mathrm{Zg}({}_{R_1}\mathscr{C})$ and $\mathrm{Zg}({}_{R_2}\mathscr{C})$ respectively.

To describe the points of the left Ziegler spectrum of R recall that a morphism $p\colon\ {}_RM \to {}_RN$ of left R-modules is a *pure-monomorphism* if the ${}_R\mathscr{C}$-morphism

$$-\otimes p\colon\ -\otimes_R M \to -\otimes_R N$$

is a monomorphism. As the tensor functor commutes with direct limits, $X \otimes_R p$ is then an Ab-monomorphism for every right R-module X_R. The left R-module ${}_RM$ is called *pure-injective* if every pure-monomorphism $p\colon\ {}_RM \to {}_RN$ is a split-monomorphism, that is, there exists a retraction $q\colon\ {}_RN \to {}_RM$ of p, with $qp = 1_M$. Pure-injective modules are closed under product/coproduct factors and arbitrary products.

Proposition 4.1 [**16**, Proposition 1.2]. *An object $E \in {}_R\mathscr{C}$ is an injective object if and only if it is isomorphic to one of the functors $-\otimes_R M$ where ${}_RM$ is a pure-injective left R-module.*

Proof. If $E \in {}_R\mathscr{C}$ is an injective object, then *a fortiori* it is coh-injective and is by Proposition 2.2 isomorphic to one of the functors $-\otimes_R M$ with ${}_RM$ a left R-module. The injective hypothesis readily implies that ${}_RM$ must in fact be pure-injective.

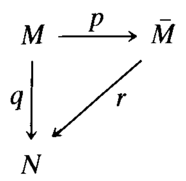
Let ${}_RM$ be pure-injective and $\mu\colon\ -\otimes_R M \to X$ a ${}_R\mathscr{C}$-monomorphism. Then μ

lifts to the injective envelope $E(X) \cong - \otimes_R N$. This lifting has the form $- \otimes_R p$ where $p\colon {}_R M \to {}_R N$ is a pure-monomorphism. As p is then a split-monomorphism, so is $- \otimes_R p$ and therefore μ is also a split-monomorphism.

A *pure-injective envelope* of a left R-module ${}_R M$ is an R-morphism $p\colon {}_R M \to {}_R \bar{M}$ such that the ${}_R\mathscr{C}$-morphism

$$- \otimes_R p\colon - \otimes_R M \to - \otimes_R \bar{M}$$

is an injective envelope in ${}_R\mathscr{C}$ of the object $- \otimes_R M$. This is equivalent to the condition that if $q\colon {}_R M \to {}_R N$ is a pure-monomorphism with ${}_R N$ pure-injective, then there is a split-monomorphism $r\colon {}_R \bar{M} \to {}_R N$ such that the diagram



commutes. The existence of pure-injective envelopes was discovered by Kiełpiński [**21**] and Warfield [**36**].

Proposition 4.1 says that the points of the left Ziegler spectrum of R are represented by the pure-injective indecomposable left R-modules. We might informally refer to a pure-injective ${}_R U$ as being a point of $\mathrm{Zg}({}_R\mathscr{C})$ when we mean to assert that $- \otimes_R U \in \mathrm{Zg}({}_R\mathscr{C})$. Note that for such a ${}_R U$, we have that $- \otimes_R U \in I(M)$ if and only if $I(U) \subseteq I(M)$. Because $\mathrm{End}_R U = \mathrm{End}_{({}_R\mathscr{C})}(- \otimes_R U)$ and $- \otimes_R U$ is an injective object in ${}_R\mathscr{C}$ one obtains the following.

PROPOSITION 4.2 [**39**]. *A pure-injective left R-module ${}_R M$ is indecomposable if and only if the endomorphism ring* $\mathrm{End}_R M$ *is local.*

EXAMPLE. Every injective indecomposable left R–module ${}_R E$ is pure-injective and hence $- \otimes_R E$ is a point of $\mathrm{Zg}({}_R\mathscr{C})$. More generally, we have the following.

PROPOSITION 4.3. *Let ${}_S M_R$ be an S-R-bimodule and ${}_S E$ an injective left S-module. The abelian group* $\mathrm{Hom}_S({}_S M_R, {}_S E)$ *equipped with the left R-module structure $(rf)(m) := f(mr)$ is a pure-injective left R-module.*

Proof. Suppose $p\colon {}_R\mathrm{Hom}_S({}_S M_R, {}_S E) \to {}_R N$ is pure. Then the S-linear map

$$M \otimes_R p\colon {}_S M \otimes_R \mathrm{Hom}_S({}_S M_R, {}_S E) \to {}_S M \otimes_R N$$

is a monomorphism. Applying the exact functor $\mathrm{Hom}_S(-, {}_S E)$ gives an epimorphism

$$(M \otimes_R p, {}_S E)\colon \mathrm{Hom}_S({}_S M \otimes_R N, {}_S E) \to \mathrm{Hom}_S({}_S M \otimes_R \mathrm{Hom}_S({}_S M_R, {}_S E), {}_S E).$$

As the tensor functor is left adjoint to the Hom functor, this epimorphism is isomorphic to the epimorphism

$$(p, {}_R\mathrm{Hom}_S({}_S M_R, {}_S E))\colon \mathrm{Hom}_R({}_R N, {}_R\mathrm{Hom}_S({}_S M_R, {}_S E)) \to \mathrm{End}_R({}_R\mathrm{Hom}_S({}_s M_R, {}_S E)).$$

A preimage of $1_{\mathrm{Hom}(M,E)}$ then gives a retraction $q\colon {}_R N \to {}_R\mathrm{Hom}_S({}_S M_{R,S} E)$.

EXAMPLE. The pure-injective indecomposable abelian groups are the following.

(1) The injective modules Q (the group of rational numbers) and, for every prime p, the Prüfer groups $Z(p^\infty)$.
(2) Every cyclic group $Z(p^n)$ of order a prime power is pure-injective because $Z(p^n) \cong (Z(p^n), Z(p^\infty))$.
(3) For every prime p, the p-adic completion $\bar{Z}_{(p)}$ of the integers is pure-injective because $\bar{Z}_{(p)} \cong (Z(p^\infty), Z(p^\infty))$.

Kaplansky [**20**] showed that this list is complete. A similar argument holds for Dedekind domains.

EXAMPLE. For a commutative noetherian ring R, Warfield [**36**] determined the pure-injective envelope ${}_R\bar{M}$ of a finitely presented (that is, finitely generated) left R-module ${}_RM$ as the completion of ${}_RM$ in the following sense. The Ω-adic topology on ${}_RM$ is the topology with a neighbourhood basis of 0 given by the submodules of the form IM as I ranges over all finite intersections of finite powers of maximal ideals. Then the Ω-adic completion $c\colon {}_RM \to {}_R\bar{M}$ is a pure-injective envelope of ${}_RM$. For example, the ring of p-adic integers $\bar{Z}_{(p)}$ is (as an abelian group) the pure-injective envelope of the localization $Z_{(p)}$ of Z at the prime p.

EXAMPLE. Let (R, m, k) be a local commutative noetherian ring with maximal ideal m and residue field k. Suppose further that R is complete in the m-adic topology. The unique simple module is k and its injective envelope $E_R(k)$ is a cogenerator in the category of left R-modules. Matlis [**24**] showed that the contravariant functor $\mathrm{Hom}_R(-, E_R(k))\colon R\text{-mod} \leftrightarrow R\text{-dcc}$ is a duality between the category R-mod of finitely generated R-modules and the category R-dcc of the modules ${}_RM$ whose lattice of submodules satisfies the descending chain condition. By Proposition 4.3, both of these categories consist of pure-injective R-modules.

EXAMPLE. Let S be a ring with centre (R, m, k) as in the previous example. Suppose furthermore that the R–module ${}_RS$ is finitely generated. If ${}_SM$ is a finitely generated S-module then it is also a finitely generated R-module and the R-isomorphism

$${}_SM \to {}_S\mathrm{Hom}_R[\mathrm{Hom}_R(M, E_R(k)), E_R(k)]$$

given by the Matlis duality is in fact an S-isomorphism. By Proposition 4.3, every finitely generated S-module is pure-injective.

EXAMPLE. An artin algebra Λ is a ring that is finitely generated as a module over an artinian centre. Every artin algebra is a finite product $\Lambda = \Lambda_1 \times \ldots \times \Lambda_n$ of artin algebras Λ_i each with a local artinian centre (R_i, m_i, k_i). As each R_i is complete, the previous example serves to show that every finitely generated indecomposable Λ-module is a point of the Ziegler spectrum

$$\mathrm{Zg}({}_\Lambda\mathscr{C}) = \bigcup_{i \leq n} \mathrm{Zg}({}_{\Lambda_i}\mathscr{C}).$$

A ring R is called *von Neumann regular* if every finitely presented left

R-module is projective. If ${}_R I \leqslant {}_R R$ is a finitely generated left ideal, then the short exact sequence $0 \to I \to R \to R/I \to 0$ splits and therefore I, a direct summand of R, is generated by an idempotent $I = Re$. Thus R is left coherent. The following proposition is essentially a result of Auslander [**1**, end of § 3].

PROPOSITION 4.4. *The following conditions on a ring R are equivalent:*

(1) *R is von Neumann regular;*

(2) *every coherent object of ${}_R\mathscr{C}$ is projective;*

(3) *every short exact sequence in* coh-$({}_R\mathscr{C})$ *splits;*

(4) *the functor ${}_R M \mapsto - \otimes_R M$ from R-*Mod *to ${}_R\mathscr{C}$ is an equivalence.*

Proof. (1)$\Rightarrow$(2). Let R be von Neumann regular and $C \in$ coh-$({}_R\mathscr{C})$. Consider a finite projective resolution of C of minimal length

$$0 \longrightarrow (M_n, -) \xrightarrow{(f_n,-)} (M_{n-1}, -) \longrightarrow \dots \longrightarrow (M_0, -) \longrightarrow C \longrightarrow 0.$$

The R-linear map f_n: $M_{n-1} \to M_n$ is an epimorphism. Because M_n is finitely presented, f_n is a split-epimorphism. But then $(f_n, -)$ is a split-monomorphism, which gives a shorter resolution, contradicting the minimal choice of the resolution. Therefore $n = 0$ and C is projective.

(2)$\Leftrightarrow$(3). Every coherent object C of ${}_R\mathscr{C}$ admits an epimorphism from a representable object. If this is a split-epimorphism, then C is projective.

(3)$\Leftrightarrow$(4). Because the functor ${}_R M \mapsto - \otimes_R M$ is always full and faithful, it is an equivalence if and only if it is dense, which is so if and only if every object X of ${}_R\mathscr{C}$ is coh-injective. This is true if and only if for every $X \in {}_R\mathscr{C}$ and $C \in$ coh-${}_R\mathscr{C}$, $\mathrm{Ext}^1_{({}_R\mathscr{C})}(C, X) = 0$, which is true if and only if every coherent C is projective.

(2)$\wedge$(4)$\Leftrightarrow$(1). From Condition (4), we know that ${}_R\mathscr{C}$ is naturally equivalent to the category R-Mod and every finitely presented object of R-Mod is therefore coherent. By Condition (2), it is projective.

So if R is von Neumann regular, then every monomorphism in R-Mod is a pure monomorphism and the Ziegler spectrum of R consists of precisely the injective indecomposable left R-modules. Every monomorphism or epimorphism in coh-$({}_R\mathscr{C})$ is split, so that for A and C in coh-$({}_R\mathscr{C})$, A is a subquotient of C if and only if $A \mid C$, that is, A is a coproduct factor of C. Thus $A \in \sqrt{C}$ if and only if there is a natural number n such that $A \mid C^{(n)}$.

Let J be a two-sided ideal of the von Neumann regular ring R and denote by $\mathscr{S}_J$ the Serre subcategory of coh-$({}_R\mathscr{C}) \cong R$-mod which consists of finitely generated summands of coproducts of finitely many copies of J, that is,

$$\mathscr{S}_J := \{P \in R\text{-mod}:\ P \mid J^{(n)} \text{ for some } n\}.$$

We claim that every Serre subcategory $\mathscr{S}$ of R-mod has this form. Let $J = t_{\mathscr{S}}(R)$, a two-sided ideal of R. If $P \in \mathscr{S}_J$, then clearly $P \in \mathscr{S}$. But if $P \in \mathscr{S}$, then, as P is projective, it is a summand of some finite power of R, $P \mid R^{(n)}$. Applying the functor $t_{\mathscr{S}}$ yields that $P \mid J^{(n)}$.

PROPOSITION 4.5. *Let R be a von Neumann regular ring and J a two-sided ideal of R. Then*

$$\mathcal{S}_J = \mathcal{S}(R/J) = \{P\colon\ (P, R/J) = 0\}.$$

Proof. First note that there are no R-linear maps $g\colon {}_RJ \to {}_R(R/J)$. For then there would be an idempotent $e \in J$ such that $g(e) \neq 0$. But $g(e) = g(e^2) = eg(e) = 0$ in R/J. Therefore $(J^{(n)}, R/J) = 0$ and so if $P \in \mathcal{S}_J$, then $(P, R/J) = 0$. On the other hand, suppose that $(P, R/J) = 0$. Let n be such that $P \mid R^{(n)}$. Then P lies in the kernel of the natural quotient map $\pi\colon R^{(n)} \to R^{(n)}/J^{(n)} \cong (R/J)^{(n)}$ and is therefore a summand of $J^{(n)}$.

THEOREM 4.6. *Let R be a von Neumann regular ring. There is an inclusion-preserving bijective correspondence between the Serre subcategories $\mathcal{S}$ of* coh-$({}_R\mathcal{C}) \cong R$-mod *and the two-sided ideals J of R given by the maps*

$$\mathcal{S} \mapsto t_{\mathcal{S}}(R) \quad \textit{and} \quad J \mapsto \mathcal{S}(R/J)$$

which are mutual inverses.

Proof. We have already noted that $J \mapsto \mathcal{S}(R/J)$ is the left inverse of $\mathcal{S} \mapsto t_{\mathcal{S}}(R)$. To show that $t_{\mathcal{S}(R/J)}(R) \geqslant J$ note that for a finitely generated left ideal ${}_RP \leqslant J$, $P \in \mathcal{S}(R/J)$ and therefore $P \leqslant t_{\mathcal{S}(R/J)}(R)$. But the module R/J is $\vec{\mathcal{S}}(R/J)$-torsion-free and therefore $t_{\mathcal{S}(R/J)}(R) \leqslant J$.

Thus the open subsets of the Ziegler spectrum of a von Neumann regular ring have the form

$$\mathcal{O}(J) := \{E \in \mathrm{Zg}(R\text{-Mod})\colon\ \mathrm{Hom}_R(J, E) \neq 0\},$$

where J is a two-sided ideal of R.

EXAMPLE. A ring R is called *indiscrete* if the topology of the left Ziegler spectrum of R is indiscrete. Equivalently, the category coh-$({}_R\mathcal{C})$ has no non-trivial Serre subcategories. By the foregoing, a von Neumann regular ring is indiscrete if and only if it is simple. Prest, Rothmaler and Ziegler [**28**, § 2.2] have constructed an example of an indiscrete ring that is not von Neumann regular.

To clarify the analogy between pure-injective and injective modules, let us point out a useful homogeneity property that pure-injective modules enjoy. Let ${}_RM$ be a left R-module and $a \in M$. Consider the R-morphism $\hat{a}\colon {}_RR \to {}_RM$ determined by the value $\hat{a}(1) = a$ and define

$$T_M(a) = \mathrm{Ker}(-\otimes \hat{a}\colon\ -\otimes_R R \to -\otimes_R M).$$

This is in some sense a 'generalized' annihilator of a in M. For example, if $g\colon {}_RM \to {}_RN$ is a morphism of R-modules such that $g(a) = b$, then

$$(-\otimes g)(-\otimes \hat{a}) = -\otimes \widehat{ga} = -\otimes \hat{b}$$

and we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_M(a) & \longrightarrow & -\otimes_R R & \longrightarrow & -\otimes_R M \\ & & \downarrow & & \downarrow{\scriptstyle -\otimes 1_R} & & \downarrow{\scriptstyle -\otimes f} \\ 0 & \longrightarrow & T_N(b) & \longrightarrow & -\otimes_R R & \longrightarrow & -\otimes_R N \end{array}$$

and the relation $T_M(a) \leqslant T_N(b)$ holds. If ${}_RN$ is pure-injective, one obtains the following converse.

PROPOSITION 4.7 [**37**, Corollary 3.3]. *Let ${}_RM$ be an arbitrary R-module, ${}_RN$ a pure-injective R-module and $a \in M$, $b \in N$. There is a morphism of R-modules f: ${}_RM \to {}_RN$ such that $f(a) = b$ if and only if $T_M(a) \leqslant T_N(b)$.*

Proof. The hypothesis gives a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_M(a) & \longrightarrow & -\otimes_R R & \longrightarrow & -\otimes_R M \\ & & \downarrow & & \downarrow{\scriptstyle -\otimes 1_R} & & \\ 0 & \longrightarrow & T_N(b) & \longrightarrow & -\otimes_R R & \longrightarrow & -\otimes_R N \end{array}$$

Because $-\otimes_R N$ is an injective object of ${}_R\mathscr{C}$, the diagram may be completed as above.

5. *Duality*

All that has been done for left R-modules may be carried out as well for right R-modules. The category (R-mod, Ab) of generalized right R-modules is then denoted by $\mathscr{C}_R$. The Ziegler spectrum $\mathrm{Zg}(\mathscr{C}_R)$ of this category is called the right Ziegler spectrum of the ring R. In this section, we give a proof of the observation, due to Auslander [**4**] and Gruson and Jensen [**16**], that there is a duality $(\text{coh-}({}_R\mathscr{C}))^{\text{op}} \cong \text{coh-}(\mathscr{C}_R)$ between the respective subcategories of the coherent objects of ${}_R\mathscr{C}$ and $\mathscr{C}_R$.

Define a functor D: $(\text{coh-}({}_R\mathscr{C}))^{\text{op}} \to \mathscr{C}_R$ on objects $C \in \text{coh-}({}_R\mathscr{C})$ as follows. Given ${}_RN \in R$-mod, we have

$$(DC)({}_RN) := \mathrm{Hom}_{({}_R\mathscr{C})}(C, -\otimes_R N)$$

If η: $B \to C$ is a $(\text{coh-}({}_R\mathscr{C}))$-morphism, then $D(\eta)_N$: $D(C)({}_RN) \to D(B)({}_RN)$ is defined to be $\mathrm{Hom}_{{}_R\mathscr{C}}(\eta, -\otimes_R N)$.

First note that the functor D: $(\text{coh-}({}_R\mathscr{C}))^{\text{op}} \to \mathscr{C}_R$ is exact. If

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is a short exact sequence in $\text{coh-}({}_R\mathscr{C})$, then because $-\otimes_R N$ is coh-injective, the sequence

$$0 \longrightarrow (C, -\otimes_R N) \xrightarrow{(\beta, -\otimes_R N)} (B, -\otimes_R N) \xrightarrow{(\alpha, -\otimes_R N)} (A, -\otimes_R N) \longrightarrow 0$$

is exact for each $N \in R$-mod. But this means the sequence

$$0 \longrightarrow DC \xrightarrow{D\beta} DB \xrightarrow{D\alpha} DA \longrightarrow 0$$

is exact.

By Yoneda's Lemma,

$$D(M_R, -)(X_R) \cong ((M_R, -), X \otimes_R -) \cong X \otimes_R M.$$

Thus $D(M_R, -) \cong M \otimes_R -$. If $C \in$ coh-$({}_R\mathscr{C})$ with projective presentation

$$(N_R, -) \to (M_R, -) \to C \to 0,$$

applying the functor D gives an exact sequence

$$0 \to DC \to M \otimes_R - \to N \otimes_R -.$$

This shows that DC is in fact a coherent object of $\mathscr{C}_R$ and therefore that the functor D has its image in coh-$(\mathscr{C}_R)$.

THEOREM 5.1 [**4**, § 7; **16**, Theorem 5.6]. *The functor* D: $(\text{coh-}({}_R\mathscr{C}))^{\text{op}} \to$ coh-$(\mathscr{C}_R)$ *defined above constitutes a duality between the categories* coh-$({}_R\mathscr{C})$ *and* coh-$(\mathscr{C}_R)$. *Furthermore, for* $M_R \in$ mod-R *and* ${}_RN \in R$-mod *we have that*

$$D(M_R, -) \cong M \otimes_R - \quad \textit{and} \quad D(- \otimes_R N) \cong ({}_RN, -).$$

Proof. First note that $D(- \otimes_R N)({}_RM) \cong (- \otimes_R N, - \otimes_R M) \cong ({}_RN, {}_RM)$ so that $D(- \otimes_R N) \cong ({}_RN, -)$. Now another functor D': $(\text{coh-}\mathscr{C}_R)^{\text{op}} \to$ coh-$({}_R\mathscr{C})$ may be defined similarly in the other direction. Both of the compositions DD' and $D'D$ are exact functors that are equivalences on the respective categories of finitely generated projective objects. Thus they are both natural equivalences.

EXAMPLE [**1**, p. 200]. Let R be a right coherent ring and M_R a finitely presented right R-module. There is then a projective resolution of M_R,

$$\cdots \xrightarrow{f_{n+1}} P_n \xrightarrow{f_n} \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M_R \longrightarrow 0$$

with every P_n finitely generated. Two complexes arise from this resolution. In coh-$({}_R\mathscr{C})$, one obtains the complex

$$0 \longrightarrow (P_0, -) \xrightarrow{(f_1,-)} (P_1, -) \longrightarrow \cdots \xrightarrow{(f_n,-)} (P_n, -) \xrightarrow{(f_{n+1},-)} \cdots$$

and in coh-$\mathscr{C}_R$, one has the dual complex

$$\cdots \xrightarrow{f_{n+1}\otimes -} P_n \otimes_R - \xrightarrow{f_n \otimes -} \cdots \longrightarrow P_1 \otimes_R - \xrightarrow{f_1 \otimes -} P_0 \otimes_R - \longrightarrow 0.$$

The homology at $(P_n, -)$ of the first complex is $\text{Ext}^n(M_R, -)$ while the dual

$$D\,\text{Ext}^n(M_R, -) = \text{Tor}_n(M_R, -)$$

is the homology of the second complex at $P_n \otimes_R -$.

We noted earlier how a ring R is von Neumann regular if and only if every short exact sequence in coh-$({}_R\mathscr{C})$ splits. By the duality D, this is clearly a left-right symmetric notion, that is, it holds for the ring R if and only if it does for the opposite ring R^{op}.

Because the category coh-$\mathscr{C}_R$ has enough projectives, the duality D gives the following.

PROPOSITION 5.2 [**4**, Lemma 7.3′]. *Every injective object of* coh-$({}_R\mathscr{C})$ *is isomorphic to one of the functors* $-\otimes_R M$ *where* ${}_RM \in R$-mod. *The category* coh-$({}_R\mathscr{C})$ *has enough injectives, that is, for every* $C \in$ coh-$({}_R\mathscr{C})$, *there is a monomorphism* $\mu\colon C \to -\otimes_R M$ *with* ${}_RM \in R$-mod.

If ${}_RM$ is a finitely presented R-module and $a \in M$, then $T_M(a)$ (defined before Proposition 4.7) is a coherent object. Using the coh-injectivity of the objects $-\otimes_R N$ where ${}_RN$ is a left R-module, one gets the following analogue of Proposition 4.7.

PROPOSITION 5.3 [**27**, Proposition 8.5]. *Let* $a \in {}_RM$, *a finitely presented* R-*module, and* $b \in {}_RN$, *an arbitrary left* R-*module. There is a morphism of* R-*modules* $g\colon {}_RM \to {}_RN$ *such that* $g(a) = b$ *if and only if* $T_M(a) \leqslant T_N(b)$.

EXAMPLE [**27**, § 11.3]. Suppose that ${}_RM$ is a finitely presented left R-module with a local endomorphism ring. The pure-injective envelope ${}_R\bar{M}$ is then indecomposable. To see this, note that the coh-$({}_R\mathscr{C})$-injective object $-\otimes_R M$ has a local endomorphism ring

$$\mathrm{End}_{\text{coh-}({}_R\mathscr{C})}(-\otimes_R M) = \mathrm{End}_R M.$$

Because the category coh-$({}_R\mathscr{C})$ has enough injectives, $-\otimes_R M$ is a uniform coh-$({}_R\mathscr{C})$-object. But then it is uniform as a $\mathscr{C}$-object and hence $E(-\otimes_R M) = -\otimes_R \bar{M}$ is indecomposable.

If furthermore ${}_RN$ is another finitely presented module such that ${}_R\bar{N} \cong {}_R\bar{M}$, then ${}_RN \cong {}_RM$. For, both $-\otimes_R N$ and $-\otimes_R M$ are essential extensions of some finitely generated, hence coherent, uniform subobject C. But then $-\otimes_R N \cong E_{\text{coh-}({}_R\mathscr{C})}(C) \cong -\otimes_R M$.

EXAMPLE. A ring R is called *Krull–Schmidt* if every finitely presented R-module is a (finite) coproduct of modules with a local endomorphism ring. This is a left-right symmetric condition on the ring R. For example, any left or right artinian ring is Krull–Schmidt. A ring S that is finitely generated as a module over a complete local noetherian centre is Krull–Schmidt, because every finitely presented (that is, finitely generated) module is pure-injective noetherian and is therefore a finite coproduct of modules with a local endomorphism ring. If R is Krull–Schmidt and ${}_RM$ is a finitely presented indecomposable R-module, then this module conforms to the previous example and therefore ${}_R\bar{M}$ is a point of the left Ziegler spectrum of R.

PROPOSITION 5.4. *Let* R *be Krull–Schmidt. The set of points having the form* ${}_R\bar{M}$ *with* ${}_RM$ *a finitely presented indecomposable module is a dense subset of the Ziegler spectrum of* R.

Proof. Let $C \in$ coh-$({}_R\mathscr{C})$. There is a monomorphism in coh-$({}_R\mathscr{C})$ of the form $\mu\colon C \to -\otimes_R M$. Now ${}_RM \cong \coprod_{i=1}^{k} {}_RM_i$ with every $-\otimes_R M_i$ a uniform object in ${}_R\mathscr{C}$. But then for some $i \leqslant k$, $(C, -\otimes_R M_i) \neq 0$ and therefore $-\otimes_R \bar{M}_i \in \mathcal{O}(C)$.

Let $\mathscr{S} \subseteq \text{coh-}({}_R\mathscr{C})$ be a Serre subcategory. It is then clear that the subcategory

$$D\mathscr{S} := \{DC \colon C \in \mathscr{S}\}$$

of coh-$\mathscr{C}_R$ is also Serre and that the restriction to $\mathscr{S}$ of the duality $D\colon (\text{coh-}_R\mathscr{C})^{\text{op}} \to \text{coh-}\mathscr{C}_R$ gives a duality $D\colon \mathscr{S}^{\text{op}} \to D\mathscr{S}$. By Theorem 3.8, the map $\mathcal{O}(\mathscr{S}) \mapsto \mathcal{O}(D\mathscr{S})$ induced on the open subsets of the left Ziegler spectrum is an inclusion-preserving bijection. Having defined the dual of a Serre subcategory, we show next that localization commutes with duality.

Let A be a coherent object of ${}_R\mathscr{C}$. If $A' \leqslant A$ is a coherent subobject, then the dualized short exact sequence in coh-$\mathscr{C}_R$,

$$0 \to D(A/A') \to DA \to DA' \to 0,$$

shows that if $A/A' \in \mathscr{S}$, then DA' is a quotient of DA by a coherent subobject in $D\mathscr{S}$. Dually, it proves that $A' \in \mathscr{S}$ if and only if $D(A/A') \leqslant DA$ is a coherent subobject such that the corresponding quotient object lies in $D\mathscr{S}$. If B is another coherent object of $\mathscr{C}$, then the function

$$(A', B/B') \mapsto (D(B/B'), DA')$$

induces an isomorphism between the partial order of pairs indexing the compatible family (1) (given at the end of § 2) and the analogous partial order corresponding to the pair (DB, DA). Now

$$D\colon \operatorname{Hom}_{{}_R\mathscr{C}}(A', B/B') \to \operatorname{Hom}_{\mathscr{C}_R}(D(B/B'), DA')$$

is an isomorphism of the related direct systems (1) which gives an isomorphism of abelian groups

$$\begin{aligned} D\colon \operatorname{Hom}_{{}_R\mathscr{C}/\vec{\mathscr{S}}}(A, B) &\cong \varinjlim \operatorname{Hom}_{{}_R\mathscr{C}}(A', B/B') \\ &\cong \varinjlim \operatorname{Hom}_{\mathscr{C}_R}(D(B/B'), DA') \\ &\cong \operatorname{Hom}_{\mathscr{C}_R/(\overrightarrow{D\mathscr{S}})}(DB, DA). \end{aligned}$$

The assignment given by $A_{\mathscr{S}} \mapsto (DA)_{D\mathscr{S}}$ on the objects and by the above for morphisms is functorial. We document all of this as follows.

THEOREM 5.5. *Let R be a ring. There is an inclusion-preserving bijective correspondence between the Serre subcategories of* coh-$({}_R\mathscr{C})$ *and those of* coh-$(\mathscr{C}_R)$ *given by*

$$\mathscr{S} \mapsto D\mathscr{S}.$$

The induced map $\mathcal{O}(\mathscr{S}) \mapsto \mathcal{O}(D\mathscr{S})$ is an isomorphism between the topologies, that is, the respective algebras of open sets, of the left and right Ziegler spectra of R. Furthermore, the duality $D\colon (\text{coh-}({}_R\mathscr{C}))^{\text{op}} \to \text{coh-}(\mathscr{C}_R)$ induces dualities between the respective subcategories $D\colon \mathscr{S}^{\text{op}} \to D\mathscr{S}$ and $D\colon (\text{coh-}({}_R\mathscr{C}/\vec{\mathscr{S}}))^{\text{op}} \to \text{coh-}(\mathscr{C}_R/(\overrightarrow{D\mathscr{S}}))$ as given by the following commutative diagram of abelian categories:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{S} & \longrightarrow & \text{coh-}({}_R\mathscr{C}) & \longrightarrow & \text{coh-}({}_R\mathscr{C}/\vec{\mathscr{S}}) & \longrightarrow & 0 \\ & & \downarrow D & & \downarrow D & & \downarrow D & & \\ 0 & \longrightarrow & D\mathscr{S} & \longrightarrow & \text{coh-}(\mathscr{C}_R) & \longrightarrow & \text{coh-}(\mathscr{C}_R/(\overrightarrow{D\mathscr{S}})) & \longrightarrow & 0 \end{array}$$

If both the left and the right Ziegler spectra of R satisfy the topological separation axiom T_0, that is, if their points are distinguished by local neighbourhood systems, then the isomorphism $\mathcal{O}(\mathcal{S}) \mapsto \mathcal{O}(D\mathcal{S})$ of open sets induces a homeomorphism

$$\mathrm{Zg}(D)\colon\ \mathrm{Zg}({}_R\mathcal{C}) \to \mathrm{Zg}(\mathcal{C}_R)$$

between the left and right Ziegler spectra of R. By this we mean that $\mathrm{Zg}(D)$ is the unique homeomorphism satisfying $\mathrm{Zg}(D)[\mathcal{O}(\mathcal{S})] = \mathcal{O}(D\mathcal{S})$.

PROPOSITION 5.6 [**40**, Lemma 2]. *Let ${}_SM_R$ be an S-R-bimodule and let ${}_SE$ be an injective S-module. Then for each $C \in$ coh-$\mathcal{C}_R$, there is an isomorphism*

$$\mathrm{Hom}_S({}_S(C, M \otimes_R -), {}_SE) \cong (DC, - \otimes_R ({}_SM_R, {}_SE))$$

natural in C.

Proof. Consider each side of the equation as a functor from coh-$\mathcal{C}_R$, with argument C, to Ab. Then each is a covariant exact functor. When $C = R \otimes_R -$ is the forgetful functor, both sides reduce to $({}_SM_R, {}_SE)$. Now every projective object $({}_RN, -)$ of coh-$\mathcal{C}_R$ has an injective resolution by finite powers of the forgetful functor. A free presentation ${}_RR^m \to {}_RR^n \to {}_RN \to 0$ of ${}_RN$ gives the exact sequence in $\mathcal{C}_R$,

$$0 \to ({}_RN, -) \to ({}_RR^n, -) \to ({}_RR^m, -).$$

By exactness, the proposition then holds for all representable functors of coh-$\mathcal{C}_R$. Applying a similar exactness argument to a projective presentation of an arbitrary $C \in$ coh-$\mathcal{C}_R$ gives the general result. Naturality also follows.

If, in addition to the hypotheses of the previous proposition, we assume that ${}_SE$ is a cogenerator, then we get the following chain of equivalences:

$$\begin{aligned} DC \in \mathcal{S}({}_R(M, E)) \quad &\Leftrightarrow \quad (DC, - \otimes_R ({}_SM_R, {}_SE)) = 0 \\ &\Leftrightarrow \quad ({}_S(C, M \otimes_R -), {}_SE) = 0 \\ &\Leftrightarrow \quad (C, M \otimes_R -) = 0 \\ &\Leftrightarrow \quad C \in \mathcal{S}(M_R). \end{aligned}$$

In short,

$$\mathcal{S}({}_R(M, E)) = D\mathcal{S}(M_R). \tag{2}$$

EXAMPLE. Let $T = \bigoplus_{p \text{ prime}} Z(p^\infty)$ denote the minimal injective cogenerator of Ab. For each prime p, the following isomorphisms are readily verified:

(1) $(Z(p^n), T) \cong (Z(p^n), Z(p^\infty)) \cong Z(p^n)$,

(2) $(Z(p^\infty), T) \cong (Z(p^\infty), Z(p^\infty)) \cong \bar{Z}_{(p)}$,

(3) $(\bar{Z}_{(p)}, T) \cong (\bar{Z}_{(p)}, Z(p^\infty)) \cong Z(p^\infty)$.

Furthermore, (Q, T) is a vector space over the rational numbers Q and so $D\mathcal{S}(Q) = \mathcal{S}(Q, T) = \mathcal{S}(Q^{(\alpha)}) = \mathcal{S}(Q)$. In this way, duality induces a homeomorphism

$$\mathrm{Zg}(D)\colon\ \mathrm{Zg}({}_Z\mathcal{C}) \to \mathrm{Zg}(\mathcal{C}_Z) = \mathrm{Zg}({}_Z\mathcal{C})$$

that fixes the rationals Q and the finite cyclic groups $Z(p^n)$ while interchanging the abelian group of p-adic integers $\bar{Z}_{(p)}$ with the Prüfer group $Z(p^\infty)$.

EXAMPLE. Let M_R be a finitely presented right R-module with local endomorphism ring S. Let ${}_SE$ be the S-injective envelope of the unique simple left S-module. Applying the isomorphism in Proposition 5.6 with $C = M \otimes_R -$ gives the isomorphism

$$\begin{aligned} {}_SE &\cong \mathrm{Hom}_S({}_SS, {}_SE) \cong \mathrm{Hom}_S({}_S(M \otimes_R -, M \otimes_R -), {}_SE) \\ &\cong ((M_R, -), - \otimes_R ({}_SM_R, {}_SE)) \cong M \otimes_R ({}_SM_R, {}_SE). \end{aligned}$$

Now we calculate the endomorphism ring of the pure-injective left R-module ${}_R({}_SM_R, {}_SE)$,

$$\begin{aligned} \mathrm{End}_R({}_SM_R, {}_SE) &\cong \mathrm{Hom}_R({}_R({}_SM_R, {}_SE), {}_R({}_SM_R, {}_SE)) \\ &\cong \mathrm{Hom}_S({}_SM \otimes_R ({}_SM_R, {}_SE), {}_SE) \cong \mathrm{Hom}_S({}_SE, {}_SE). \end{aligned}$$

This shows that the endomorphism ring is local and hence that ${}_R({}_SM_R, {}_SE)$ is a pure-injective indecomposable left R-module.

In general, we let $\mathrm{Zg}(D)$: $\mathrm{Zg}({}_R\mathscr{C}) \to \mathrm{Zg}(\mathscr{C}_R)$ be the partial function whose domain consists of those $U \in \mathrm{Zg}({}_R\mathscr{C})$ uniquely determined by a local system of basic open neighbourhoods and having the additional property that the system of basic open neighbourhoods in $\mathrm{Zg}(\mathscr{C}_R)$ dual to the system of U, determines a unique point V. Then $\mathrm{Zg}(D)(U) = V$. If ${}_RM$ is a finitely presented left R-module with local endomorphism ring S, the previous example computes the value of $\mathrm{Zg}(D)({}_R\bar{M})$ as $\mathrm{Hom}_S({}_RM_S, E_S)_R$ when it is defined.

EXAMPLE. Consider the right R-module R_R and let ${}_RE$ be an injective cogenerator for R-Mod. Then ${}_R({}_RR_R, {}_RE) \cong {}_RE$ and equation (2) becomes $D\mathscr{S}(R_R) = \mathscr{S}({}_RE)$. We can describe the objects of $\mathscr{S}(R_R)$ as follows. Take a projective presentation of $C \in \text{coh-}\mathscr{C}_R$,

$$({}_RN, -) \xrightarrow{(f,-)} ({}_RM, -) \longrightarrow C \longrightarrow 0$$

and apply the functor $\mathrm{Hom}_{({}_R\mathscr{C})}(?, (R \otimes_R -))$. By Yoneda's Lemma, this gives an exact sequence $0 \to (C, R \otimes_R -) \to M_R \xrightarrow{f} N_R$ of right R-modules. Now $C \in \mathscr{S}(R_R)$ if and only if $(C, R \otimes_R -) = 0$, which is so if and only if f is an R-monomorphism. Thus

$$\mathscr{S}(R_R) = \{C \cong \mathrm{Coker}(f, -)\colon\ f \text{ an } R\text{-monomorphism}\}.$$

Aplying the duality D gives $\mathscr{S}({}_RE) = \{DC \cong \mathrm{Ker}(f \otimes -)\colon f \text{ an } R\text{-monomorphism}\}$.

PROPOSITION 5.7. *Let ${}_RE$ be an injective cogenerator for R-Mod. If ${}_RE'$ is an injective left R-module, then $I({}_RE') \subseteq I({}_RE)$. Moreover $I({}_RE)$ is the closure in the left Ziegler spectrum of R of the indecomposable injective left R-modules.*

Proof. Let $C \in \text{coh-}({}_R\mathscr{C})$ and take a $(\text{coh-}({}_R\mathscr{C}))$-injective copresentation of C,

$$0 \longrightarrow C \longrightarrow - \otimes_R M \xrightarrow{-\otimes f} - \otimes_R N.$$

If ${}_RX$ is a left R-module, then applying the exact functor $\mathrm{Hom}(?, -\otimes_R X)$ gives the exact sequence

$$({}_RN, {}_RX) \xrightarrow{(f, X)} ({}_RM, {}_RX) \longrightarrow (C, -\otimes_R X) \longrightarrow 0.$$

Thus $(C, -\otimes_R X) \cong \mathrm{Coker}(f, X)$. If $C \in \mathscr{S}({}_RE)$, then f must be an R-monomorphism and so for any injective left R-module ${}_RE'$, we have that $(C, -\otimes_R E') \cong \mathrm{Coker}(f, E') = 0$. Thus $I(E') \subseteq I({}_RE)$. In particular, if ${}_RU$ is an indecomposable injective left R-module, then $I({}_RU) \subseteq I({}_RE)$ implies that $-\otimes_R U \in I({}_RE)$.

To show that $I({}_RE)$ is the closure of the set of injective indecomposables ${}_RU$, suppose to the contrary, that $V \in I(E) \cap \mathcal{O}(C)$, but that for each of the injective U, $-\otimes_R U \notin \mathcal{O}(C)$. Now $(C, -\otimes_R U) \cong \mathrm{Coker}(f, {}_RU) = 0$ for all ${}_RU$. Because the injective indecomposables ${}_RU$ form a cogenerating set, f must be an R-monomorphism. But then $C \in \mathscr{S}({}_RE)$, contradicting $I({}_RE) \cap \mathcal{O}(C) \neq \varnothing$.

Prest, Rothmaler and Ziegler have shown [**28**, Corollary 4.4] that the ring R is left coherent if and only if $I({}_RE)$ consists exclusively of indecomposable injectives.

6. *Finite matrix subgroups*

In this section, we shall consider exact representations of the category coh-$\mathscr{C}$ into module categories. Let $M \in \mathscr{C}$ be a coh-injective object with endomorphism ring $S = \mathrm{End}_{\mathscr{C}} M$. Then the exact contravariant functor

$$\mathrm{Hom}_{\mathscr{C}}(-, M)\colon \text{coh-}\mathscr{C} \to S\text{-Mod}$$

is such a representation. If $C \in \mathscr{C}$ is a coherent object, the S-submodules of ${}_S(C, M)$ corresponding to coherent quotient objects of C are called finite matrix subgroups of ${}_S(C, M)$. They were introduced by Gruson and Jensen [**15**] and Zimmermann [**38**]. Through the work of Baur [**6**], they have become the central motif of the model theory of modules. If ${}_RX$ is a left R-module with endomorphism ring $S = \mathrm{End}_R X = \mathrm{End}_{({}_R\mathscr{C})}(-\otimes_R X)$, finite matrix subgroups of the S-module ${}_S(-\otimes_R R, -\otimes_R X) \cong {}_SX$ provide some control over the complexity of the localized category ${}_R\mathscr{C}({}_RX)$ in terms of the lattice of S-submodules of ${}_SX$. We shall refer to an S-submodule of ${}_SX$ as an *endosubmodule* of ${}_RX$.

Let C be a coherent object of $\mathscr{C}$. Because the category coh-$\mathscr{C}$ is abelian, the coherent subobjects of C form a modular lattice $L_{\text{coh-}\mathscr{C}}(C)$ [**34**, Proposition IV.5.3] with maximum and minimum elements. All lattices mentioned in the sequel will be modular with maximum and minimum elements which lattice morphisms are presumed to preserve. We begin with the observation that localization induces such a lattice morphism.

Proposition 6.1. *Let F: $\mathscr{A} \to \mathscr{B}$ be an exact functor of abelian categories and $A \in \mathscr{A}$. The map*

$$L(F)\colon L_{\mathscr{A}}(A) \to L_{\mathscr{B}}(FA)$$

this sends the subobject μ: $C \leqslant A$ *to the subobject* $F(\mu)$: $F(C) \leqslant F(A)$ *is a morphism of lattices.*

Proof. Let β: $B \leqslant A$ and γ: $C \leqslant A$ be subobjects of A. Then $B + C = \operatorname{Im}(\beta \amalg \gamma)$ and so

$$F(B + C) = F(\operatorname{Im}(\beta \amalg \gamma)) = \operatorname{Im}(F(\beta) \amalg F(\gamma)) = F(B) + F(C).$$

A similar argument in the opposite category proves the dual statement.

Let C be a coherent object of $\mathscr{C}$ and $\mathscr{S} \subseteq$ coh-$\mathscr{C}$ a Serre subcategory. The proposition implies that localization at $\mathscr{S}$ induces a lattice morphism

$$L(-)_{\mathscr{S}}\colon\ L_{\text{coh-}\mathscr{C}}(C) \to L_{\text{coh-}\mathscr{C}/\vec{\mathscr{S}}}(C_{\mathscr{S}}).$$

The next proposition describes the morphism intrinsically.

Proposition 6.2. *The lattice morphism* $L(-)_{\mathscr{S}}$: $L_{\text{coh-}\mathscr{C}}(C) \to L_{\text{coh-}\mathscr{C}/\vec{\mathscr{S}}}(C_{\mathscr{S}})$ *is the quotient morphism of the lattice* $L_{\text{coh-}\mathscr{C}}(C)$ *modulo the congruence*

$$A + B/(A \cap B) \in \mathscr{S}.$$

Proof. By the coherent version of Proposition 2.14, the lattice morphism $L(-)_{\mathscr{S}}$: $L_{\text{coh-}\mathscr{C}}(C) \to L_{\text{coh-}\mathscr{C}/\vec{\mathscr{S}}}(C_{\mathscr{S}})$ is surjective. Now note that $A_{\mathscr{S}} \cong B_{\mathscr{S}}$ if and only if $A_{\mathscr{S}} + B_{\mathscr{S}}/(A_{\mathscr{S}} \cap B_{\mathscr{S}}) \cong (A + B)_{\mathscr{S}}/(A \cap B)_{\mathscr{S}} = 0$. By the exactness of localization, that is equivalent to $[A + B/(A \cap B)]_{\mathscr{S}} = 0$ which is equivalent to $A + B/(A \cap B) \in \mathscr{S}$.

Let $C \in$ coh-$\mathscr{C}$ and let $M \in \mathscr{C}$ be a coh-injective object. If $A \in L_{\text{coh-}\mathscr{C}}(C)$, and $S = \operatorname{End}_{\mathscr{C}} M$, then the short exact sequence

$$0 \to A \to C \to C/A \to 0$$

gives rise to a short exact sequence

$$0 \to {}_S(C/A, M) \to {}_S(C, M) \to {}_S(A, M) \to 0$$

of left S-modules. In this manner, we identify ${}_S(C/A, M)$ with an element of $L_S((C, M))$, the lattice of S-submodules of ${}_S(C, M)$. The map defined by $A \mapsto (C/A, M)$ will be denoted by $\delta_{(C,M)}$: $L_{\text{coh-}\mathscr{C}}(C) \to L_S((C, M))$. A *finite matrix subgroup* of (C, M) is defined to be any S-submodule of ${}_S(C, M)$ that is in the image of $\delta_{(C,M)}$.

Example. Let R be a ring and C a coherent object of ${}_R\mathscr{C}$. If ${}_RM$ is a finitely presented left R-module, then the finite matrix subgroups of $DC({}_RM) = (C, - \otimes_R M)$ have the form $D(C/A)(M)$ as C/A ranges over the coherent quotient objects of C. Thus the finite matrix subgroups of $(C, - \otimes_R M)$ are precisely the subgroups of the form $B(M)$ where $B \in L_{\text{coh-}\mathscr{C}_R}(DC)$.

Proposition 6.3. *Let* C *be a coherent object of* $\mathscr{C}$ *and* $M \in \mathscr{C}$ *a coh-injective object with* $S = \operatorname{End}_{\mathscr{C}} M$. *The map* $\delta_{(C,M)}$: $L_{\text{coh-}\mathscr{C}}(C) \to L_S((C, M))$ *is an anti-morphism of lattices. It induces an anti-embedding of* $L_{\text{coh-}\mathscr{C}(M)}(C_{\mathscr{S}(M)})$ *into* $L_S((C, M))$.

Proof. The first statement follows from Proposition 6.1 and the fact that the functor $\mathrm{Hom}_{\mathscr{C}}(-, M)$ is exact. To prove the second, let $A, B \leq C$ be coherent. We have the inclusion $(C/(A+B), M) \leq (C/(A \cap B), M)$ in $L_S((C, M))$ and it is clear that this inclusion is proper if and only if $A + B/(A \cap B) \notin \mathscr{S}(M)$. Thus for $A, B \in L_{\mathrm{coh}\text{-}\mathscr{C}}(C)$, it follows that $\delta_{(C,M)}(A) = \delta_{(C,M)}(B)$ if and only if $\delta_{(C,M)}(A+B) = \delta_{(C,M)}(A \cap B)$, which is true if and only if $A + B/(A \cap B) \in \mathscr{S}(M)$. By Proposition 6.2, an anti-embedding of $L_{\mathrm{coh}\text{-}\mathscr{C}(M)}(C_{\mathscr{S}(M)})$ into $L_S((C, M))$ is induced.

EXAMPLE. Suppose that $\mathscr{C} = {}_R\mathscr{C}$ for some ring R and let $C = - \otimes_R R$. If $- \otimes_R X$ is a coh-injective object of ${}_R\mathscr{C}$ with endomorphism ring $S = \mathrm{End}_R X$, the finite matrix subgroups of ${}_S(- \otimes_R R, - \otimes_R X) \cong {}_SX$ may be identified with certain endosubmodules of ${}_RX$. Let us describe then the finite matrix subgroups of ${}_RX$. For simplicity, abbreviate $\delta_{(-\otimes R, -\otimes X)}$ to δ_X. If $A \leq - \otimes_R R$ is coherent, there is a monomorphism $(- \otimes_R R)/A \to - \otimes_R M$ for some finitely presented left R-module ${}_RM$. From the exact sequence

$$0 \longrightarrow A \longrightarrow - \otimes_R R \xrightarrow{-\otimes f} - \otimes_R M$$

we see that $A = T_M(f(1))$, and by the coh-injectivity of $- \otimes_R X$,

$$\delta_X(A) = (\mathrm{Im}\,(-\otimes f), - \otimes_R X) = \{y \in X\colon\ A = T_M(f(1)) \leq T_X(y)\}.$$

We shall often apply the following special case of Proposition 6.3.

PROPOSITION 6.4. *Let ${}_RX$ be an R-module and $S = \mathrm{End}_R X$. There is an anti-embedding of the lattice $L_{\mathrm{coh}\text{-}\mathscr{C}(X)}[(- \otimes_R R)_{\mathscr{S}(X)}]$ into the lattice $L_S(X)$ of endosubmodules of ${}_RX$.*

If $M \in \mathscr{C}$ is coh-injective and $\mathscr{S} \subseteq \mathscr{S}(M)$ is a Serre subcategory, then M is also coh-injective as an object of $\mathscr{C}/\vec{\mathscr{S}}$ and the respective endomorphism rings coincide, $S = \mathrm{End}_{\mathscr{C}} M = \mathrm{End}_{\mathscr{C}/\vec{\mathscr{S}}} M$. As localization is a left adjoint, the isomorphism $\mathrm{Hom}_{\mathscr{C}}(C, M) \cong \mathrm{Hom}_{\mathscr{C}/\vec{\mathscr{S}}}(C_{\mathscr{S}}, M)$ is, by naturality, an S-module isomorphism. Thus the lattices $L_S(C, M)$ and $L_S(C_{\mathscr{S}}, M)$ are isomorphic. The following proposition shows that this isomorphism, which we denote by $(-)_{\mathscr{S}}\colon L_{\mathscr{S}}(C, M) \to L_S(C_{\mathscr{S}}, M)$, preserves finite matrix subgroups.

PROPOSITION 6.5. *Let $M \in \mathscr{C}$ be coh-injective with endomorphism ring $S = \mathrm{End}_{\mathscr{C}} M$. If $\mathscr{S} \subseteq \mathscr{S}(M)$ is a Serre subcategory of* coh-$\mathscr{C}$ *and $C \in$ coh-$\mathscr{C}$, then the following diagram of lattice morphisms commutes:*

$$\begin{array}{ccc} L_{\mathrm{coh}\text{-}\mathscr{C}}(C) & \xrightarrow{L(-)_{\mathscr{S}}} & L_{\mathrm{coh}\text{-}\mathscr{C}/\vec{\mathscr{S}}}(C_{\mathscr{S}}) \\ \Big\downarrow{\scriptstyle \delta_{(C,M)}} & & \Big\downarrow{\scriptstyle \delta_{(C_{\mathscr{S}},M)}} \\ L_S(C, M) & \xrightarrow[(-)_{\mathscr{S}}]{} & L_S(C_{\mathscr{S}}, M) \end{array}$$

Proof. Let $A \leq C$ be a coherent subobject. On the one hand, $(-)_{\mathscr{S}}\delta_{(C,M)}(A) = ((C/A)_{\mathscr{S}}, M)$ while $\delta_{(C_{\mathscr{S}},M)}L(-)_{\mathscr{S}}(A) = ((C_{\mathscr{S}}/A_{\mathscr{S}}), M)$. Since the quotient $C_{\mathscr{S}}/A_{\mathscr{S}}$ is taken in $\mathscr{C}/\vec{\mathscr{S}}$, the two are equal.

Let $M \in \mathscr{C}$ be an injective object with endomorphism ring $S = \mathrm{End}_{\mathscr{C}} M$. We turn our attention now to finite matrix subgroups that are cyclic as S-modules. If $\eta \in {}_S(C, M)$, then the cyclic subgroup $S\eta$ may be computed using injectivity as

$$S\eta = {}_S(\mathrm{Im}\ \eta, M) = (C/\mathrm{Ker}\ \eta, M).$$

Writing $\mathrm{Ker}\ \eta = \sum_{i \in I} A_i$ as a direct union of coherent subobjects of C shows that

$$S\eta = \left(C\Big/\Big(\sum_{i\in I} A_i\Big), M\right) = \bigcap_{i \in I} (C/A_i, M) = \bigcap_{i\in I} \delta_{(C,M)}(A_i).$$

This proves the first part of the next proposition.

Proposition 6.6 [**8**, Lemma 4.1; **27**, Exercise 11.3; **37**, Corollary 3.3(1)]. *If M is an injective object of $\mathscr{C}$ with endomorphism ring $S = \mathrm{End}_{\mathscr{C}} M$, then every cyclic S-submodule of ${}_S(C, M)$ is an intersection of finite matrix subgroups. If M is an injective object of* coh-$\mathscr{C}$*, then every finitely generated S-submodule of ${}_S(C, M)$ is a finite matrix subgroup.*

Proof. Let M be an injective object of coh-$\mathscr{C}$. If $\eta \in {}_S(C, M)$, then $\mathrm{Ker}\ \eta$ is a coherent subobject of C and, as above,

$$S\eta = {}_S(\mathrm{Im}\ \eta, M) = (C/\mathrm{Ker}\ \eta, M) = \delta_{(C,M)}(\mathrm{Ker}\ \eta).$$

Thus every cyclic S-submodule of ${}_S(C, M)$ is a finite matrix subgroup. As the finite matrix subgroups of (C, M) are closed under finite sums, every finitely generated S-submodule of ${}_S(C, M)$ is a finite matrix subgroup.

7. *Examples of Serre subcategories*

As in [**27**, Chapter 10], we shall consider in this section Serre subcategories that arise from lattice-theoretic considerations. Recall that if ${}_R M$ is a left R-module, we denote by ${}_R\mathscr{C}(M)$ the quotient category ${}_R\mathscr{C}/\vec{\mathscr{S}}(M)$. We shall give examples of R-modules ${}_R M$ such that the objects of coh-$({}_R\mathscr{C}(M))$ are in a lattice-theoretic sense well behaved.

An object $X \in \mathscr{C}$ is called *noetherian* if every subobject of X is finitely generated. For $C \in$ coh-$\mathscr{C}$, this is equivalent to the ascending chain condition on $L_{\text{coh-}\mathscr{C}}(C)$. It is clear that the noetherian coherent objects form a Serre subcategory acc-$\mathscr{C} \subseteq$ coh-$\mathscr{C}$. The category $\mathscr{C}$ is called *locally noetherian* if the equality acc-$\mathscr{C}$ = coh-$\mathscr{C}$ holds. In that case, every finitely generated object of $\mathscr{C}$ is coherent. For example, if R is a left noetherian ring, then the category R-Mod is locally noetherian. The mysterious rings for which the category ${}_R\mathscr{C}$ is locally noetherian are called *left pure-semisimple.*

Proposition 7.1 [**31**]. *The following are equivalent for a locally coherent Grothendieck category $\mathscr{C}$.*

(1) *The category $\mathscr{C}$ is locally noetherian.*

(2) *Every* coh-*injective object of $\mathscr{C}$ is injective.*

(3) *The injective objects of $\mathscr{C}$ are closed under coproducts.*

Proof. (1) ⇒ (2). Suppose that M is a coh-injective object and that $M \leqslant E(M)$ is a proper extension. There exists a non-trivial essential (hence non-split) extension $M \leqslant X \leqslant E(M)$ such that X/M is finitely generated and therefore, by hypothesis, coherent. This contradicts $\mathrm{Ext}^1_{\mathscr{C}}(X/M, M) = 0$.

(2) ⇒ (3). The coh-injective objects of $\mathscr{C}$ are closed under coproducts.

(3) ⇒ (1). See [**34**, Proposition V.4.3].

Suppose that $_RM$ is an R-module such that $_R\mathscr{C}(M)$ is locally noetherian. By Proposition 7.1, the object $- \otimes_R M$ is injective in $_R\mathscr{C}(M)$ and therefore it is injective in $_R\mathscr{C}$. Thus $_RM$ is a pure-injective R-module [**13**, **38**]. So if R is left pure-semisimple, then Proposition 7.1 implies that every left R-module is pure-injective and that the left Ziegler spectrum $\mathrm{Zg}(_R\mathscr{C})$ of R consists of all the indecomposable left R-modules. More generally, the Ziegler spectrum $\mathrm{Zg}(\mathscr{C})$ of a locally noetherian Grothendieck category $\mathscr{C}$ consists of the indecomposable coh-injective objects.

Proposition 7.2. *Let $_RM$ be an R-module which satisfies the descending chain condition on endosubmodules. Then the category $_R\mathscr{C}(M)$ is locally noetherian.*

Proof. Localizing the equation $\sqrt{(- \otimes_R R)} = \text{coh-}(_R\mathscr{C})$ at $\mathscr{S}(M)$ gives

$$\sqrt{((- \otimes_R R)_{\mathscr{S}(M)})} = \text{coh-}(_R\mathscr{C}(M)).$$

So it suffices to prove that $(- \otimes_R R)_{\mathscr{S}(M)} \in \text{acc-}(_R\mathscr{C}(M))$. But this is immediate from Proposition 6.4.

A dual version of Proposition 7.2 is obtained similarly. Define dcc-$\mathscr{C}$ to be the Serre subcategory consisting of those coherent objects $C \in \mathscr{C}$ for which $L_{\text{coh-}\mathscr{C}}(C)$ satisfies the descending chain condition.

Proposition 7.3. *Let $_RM$ be an R-module which satisfies the ascending chain condition on endosubmodules. Then the category* $\text{coh-}(_R\mathscr{C}(M)) = \text{dcc-}(_R\mathscr{C}(M))$.

Example. If R is a right noetherian ring, then the left R-module $_RR$ satisfies the ascending chain condition on endosubmodules. Thus $\text{coh-}(_R\mathscr{C}(R)) = \text{dcc-}(_R\mathscr{C}(R))$. If E_R is an injective cogenerator, then, by duality, the category $\mathscr{C}_R(E)$ is locally noetherian.

Example. A ring R is called a *noetherian algebra* if it is finitely generated as a module over a noetherian centre. If $_RM$ is finitely presented, that is, finitely generated, then it is clear that M is finitely generated and hence a noetherian module over the centre of R. The module $_RM$ must therefore satisfy the ascending chain condition on endosubmodules.

Proposition 7.4 [**40**, Observation 8]. *Let $_RM$ be a finitely presented R-module with endomorphism ring $S = \mathrm{End}_R\, M$ and let $C \in {_R\mathscr{C}}$ be a coherent object. If* $\text{coh-}(_R\mathscr{C}(M)) = \text{dcc-}(_R\mathscr{C}(M))$, *the S-module $_S(C, M)$ is noetherian. In particular, every S-submodule is a finite matrix subgroup.*

Proof. By Proposition 6.6, every finitely generated S-submodule is a finite matrix subgroup. By hypothesis, these satisfy the ascending chain condition. Therefore every S-submodule of ${}_S(C, M)$ is finitely generated.

Let fin-$\mathscr{C} \subseteq$ coh-$\mathscr{C}$ denote the category of those coherent objects C for which the lattice $L_{\text{coh-}\mathscr{C}}(C)$ has a composition series. In other words, fin-$\mathscr{C}$ consists of the coherent objects of finite length. This is the Serre subcategory fin-$\mathscr{C}$ = acc-$\mathscr{C} \cap$ dcc-$\mathscr{C}$. The category $\mathscr{C}$ is called *locally finite* if coh-$\mathscr{C}$ = fin-$\mathscr{C}$. For example, if $C \in$ fin-$\mathscr{C}$, then $\mathscr{S} = \sqrt{C} \subseteq$ fin-$\mathscr{C}$ and so the category $\vec{\mathscr{S}}$ is locally finite.

PROPOSITION 7.5. *Let $C \in$ fin-$\mathscr{C}$ and $M \in \mathscr{C}$ a coh-injective object with $S = \operatorname{End}_R M$. Then every S-submodule of ${}_S(C, M)$ is a finite matrix subgroup.*

Proof. Let $\mathscr{S} = \sqrt{C}$ and note that the $\vec{\mathscr{S}}$-object $t_{\mathscr{S}}(M)$ is coh-injective. For, if $S \in \mathscr{S}$, consider the beginning of the long exact sequence

$$0 \to (S, t_{\mathscr{S}}(M)) \to (S, M) \to (S, M/t_{\mathscr{S}}(M)) \to \operatorname{Ext}^1(S, t_{\mathscr{S}}(M)) \to \operatorname{Ext}^1(S, M) \to \dots .$$

As M is coh-injective, $\operatorname{Ext}^1(S, M) = 0$ and as $M/t_{\mathscr{S}}(M)$ is $\vec{\mathscr{S}}$-torsion-free, $(S, t_{\mathscr{S}}(M)) = 0$. Thus $\operatorname{Ext}^1(S, t_{\mathscr{S}}(M)) = 0$ for every $S \in \mathscr{S}$ and therefore $t_{\mathscr{S}}(M)$ is a coh-injective object of $\vec{\mathscr{S}}$.

Now ${}_S(C, M) = {}_S(C, t_{\mathscr{S}}(M))$, so we need to prove that every S-submodule of ${}_S(C, t_{\mathscr{S}}(M))$ is a finite matrix subgroup. As $\vec{\mathscr{S}}$ is locally finite, $t_{\mathscr{S}}(M)$ is in fact an injective object of $\vec{\mathscr{S}}$. By Proposition 6.6, every cyclic S-submodule is an intersection of finite matrix subgroups. By the ascending chain condition in $L_{\mathscr{S}}(C)$ and Proposition 6.3, every cyclic S-submodule is a finite matrix subgroup. By the descending chain condition in $L_{\mathscr{S}}(C)$ and Proposition 6.3, every S-submodule is a finite matrix subgroup.

Call a left R-module ${}_R M$ *endofinite* if M has finite length as a module over its endomorphism ring $\operatorname{End}_R M$.

PROPOSITION 7.6. *Let ${}_R M$ be a left R-module. The category ${}_R\mathscr{C}(M)$ is locally finite if and only if the module ${}_R M$ is endofinite.*

Proof. If ${}_R M$ is endofinite, then ${}_R\mathscr{C}(M)$ is locally finite by Proposition 6.4. If ${}_R\mathscr{C}(M)$ is locally finite, then $(-\otimes_R R)_{\mathscr{S}(M)} \in$ fin-$({}_R\mathscr{C}(M))$. Let $S = \operatorname{End}_R M$. By the previous proposition, every S-submodule of

$${}_S((-\otimes_R R)_{\mathscr{S}(M)}, -\otimes_R M) = {}_S(-\otimes_R R, -\otimes_R M) = {}_S M$$

is a finite matrix subgroup. By Proposition 6.4, the lattice of finite matrix subgroups of ${}_S M$ has a composition series.

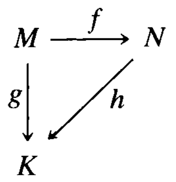
Obviously fin-$\mathscr{C} = \sqrt{\{S \in \text{coh-}\mathscr{C}\colon S \text{ simple}\}}$ so that in the Ziegler spectrum of $\mathscr{C}$ we have

$$\mathcal{O}(\text{fin-}\mathscr{C}) = \bigcup \{\mathcal{O}(S)\colon S \in \text{coh-}\mathscr{C} \text{ is simple}\}.$$

If $U \in \mathcal{O}(\text{fin-}\mathscr{C})$, then there is a coherent simple object S such that $(S, U) \neq 0$. It follows that $U = E(S)$ and that $\mathcal{O}(S) = \{U\}$. Thus U is an isolated point of $\operatorname{Zg}(\mathscr{C})$.

Consequently, any dense subset of $\mathrm{Zg}(\mathscr{C})$ contains $\mathcal{O}(\text{fin-}\mathscr{C})$. For example, $\mathcal{O}(\text{fin-}\mathscr{C}) \subseteq \max(\mathscr{C})$. If R is Krull–Schmidt, then by Proposition 5.4, every point of $\mathcal{O}(\text{fin-}({}_R\mathscr{C}))$ is of the form $-\otimes_R \bar{M}$ where ${}_RM$ is a finitely presented indecomposable left R-module.

To give a criterion for when $-\otimes_R \bar{M} \in \mathcal{O}(\text{fin-}({}_R\mathscr{C}))$ recall from [**2**] that a morphism $f\colon {}_RM \to {}_RN$ in R-mod which is not a split-monomorphism is called *left almost split* if any morphism $g\colon {}_RM \to {}_RK$ in R-mod that is not a split-monomorphism factors through f, that is, there exists a morphism $h\colon {}_RN \to {}_RK$ such that the following diagram commutes:



Proposition 7.7 [**2**]. *Let ${}_RM$ be a finitely presented R-module. The following are equivalent:*

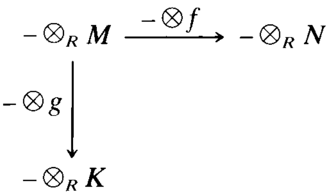
(1) $-\otimes_R \bar{M} \in \mathcal{O}(\text{fin-}({}_R\mathscr{C}))$;

(2) $-\otimes_R M$ *is essential over a simple subobject* S;

(3) *there is a left almost split morphism* $f\colon {}_RM \to {}_RN$ *in* R-mod.

Proof. (1)$\Leftrightarrow$(2). If $-\otimes_R \bar{M} \in \mathcal{O}(\text{fin-}({}_R\mathscr{C}))$, there is a simple subobject $S \leqslant (-\otimes_R \bar{M}) = E(-\otimes_R M)$. Thus $-\otimes_R M$ is essential over S. Conversely, if $S \leqslant -\otimes_R M$ is an essential extension, then S is coherent and $-\otimes_R \bar{M}$ is indecomposable. Furthermore, $(S, -\otimes_R \bar{M}) \neq 0$ and so $-\otimes_R \bar{M} \in \mathcal{O}(\text{fin-}({}_R\mathscr{C}))$.

(2)$\Leftrightarrow$(3). Consider the diagram



That $-\otimes f$ is not a split-monomorphism is tantamount to $\mathrm{Ker}(-\otimes f) \neq 0$. That every $-\otimes g$ that is not a split-monomorphism factors through $-\otimes f$ is tantamount to $\mathrm{Ker}(-\otimes f)$ being contained in every non-zero subobject of $-\otimes_R M$ (one can see this by using the existence of injectives in $\text{coh-}({}_R\mathscr{C})$). Thus $f\colon {}_RM \to {}_RN$ is left almost split if and only if $\mathrm{Ker}(-\otimes f)$ is simple and $-\otimes_R M$ is an essential extension of it.

A similar result holds for the category R-Mod.

Proposition 7.8 [**8**, Theorem 2.3]. *Let ${}_RM$ be a pure-injective indecomposable R-module. Then $-\otimes_R M \in \max({}_R\mathscr{C})$ if and only if there is a left almost split morphism $f\colon {}_RM \to {}_RN$ in the category* R-Mod.

Example. The maximal Ziegler spectrum $\max({}_Z\mathscr{C})$ of ${}_Z\mathscr{C}$ consists of the torsion pure-injective indecomposable abelian groups. Thus the self-homeomorphism $\mathrm{Zg}(D)$ does not preserve the maximal Ziegler spectrum. If p is prime, the basic open set $\mathcal{O}(-\otimes_Z Z(p))$ consists of the p-torsion pure-injective indecomposable abelian groups. The open set

$$\mathcal{O}(-\otimes_Z Z(p)) \cup \mathcal{O}((Z(p), -)) = \mathcal{O}(-\otimes_Z Z(p) \amalg (Z(p), -))$$

contains only one additional point, the abelian group of p-adic integers. These two distinct basic open subsets agree on $\max({}_Z\mathcal{C})$, which shows that Condition (8) of Corollary 3.12 cannot be strengthened to $\max(\mathcal{C})$.

Example [**27**, §13.1]. Let Λ be an artin algebra. Every finitely generated Λ-module ${}_\Lambda M$ is pure-injective. Auslander and Reiten [**5**] proved that every indecomposable Λ-module admits a left almost split morphism. Thus $\mathcal{O}(\text{fin-}({}_\Lambda\mathcal{C}))$ consists of the finitely generated indecomposable Λ-modules. As this set is dense, these are precisely the isolated points of the Ziegler spectrum. Furthermore, we have that

$$\mathcal{O}(\text{fin-}({}_\Lambda\mathcal{C})) = \max({}_\Lambda\mathcal{C}).$$

To see this, suppose that $S \in {}_\Lambda\mathcal{C}$ is simple. We want to show that $E(S) \in \mathcal{O}(\text{fin-}({}_\Lambda\mathcal{C}))$. There is a finitely generated indecomposable right Λ-module M_Λ such that $S(M_\Lambda) \neq 0$. By Yoneda's Lemma, there is a non-zero morphism $(M_\Lambda, -) \to S$. We will use duality to prove S is coherent. We know that M_Λ is isolated in $\mathrm{Zg}(\mathcal{C}_\Lambda)$ by some coherent simple object $S' \in {}_\Lambda\mathcal{C}$, $\mathcal{O}(S') = \{M \otimes_\Lambda -\}$. Applying duality to the monomorphism $S' \leqslant M \otimes_\Lambda -$ in coh-$\mathcal{C}_\Lambda$ implies that $S = DS' \in \text{coh-}({}_\Lambda\mathcal{C})$ because $(M_\Lambda, -) = D(M \otimes_\Lambda -)$ is a local functor.

A ring R is said to be of *finite representation type* if R is left artinian with just finitely many finitely generated indecomposable left R-modules. Auslander [**2**] showed that this is equivalent to the condition that $-\otimes_R R \in \text{fin-}({}_R\mathcal{C})$, that is, that the category ${}_R\mathcal{C}$ is locally finite. By Theorem 3.8, this is equivalent to the equation $\mathrm{Zg}({}_R\mathcal{C}) = \mathcal{O}(-\otimes_R R) = \mathcal{O}(\text{fin-}({}_R\mathcal{C}))$. Applying the duality D to the relation $-\otimes_R R \in \text{fin-}({}_R\mathcal{C})$ gives that $R \otimes_R - \in D[\text{fin-}({}_R\mathcal{C})] = \text{fin-}(\mathcal{C}_R)$. Thus finite representation type is a left-right symmetric notion.

Proposition 7.9 [**3**; **27**, Corollary 13.4]. *Let Λ be an artin algebra. Then Λ is not of finite representation type if and only if there is a* (*pure-injective*) *indecomposable left Λ-module which is not finitely generated.*

Proof. The artin algebra Λ is not of finite representation type if and only if $-\otimes_\Lambda \Lambda \notin \text{fin-}({}_\Lambda\mathcal{C})$. This is so if and only if, by Theorem 3.8, the inclusion $\mathcal{O}(\text{fin-}({}_\Lambda\mathcal{C})) \subseteq \mathcal{O}(-\otimes_\Lambda \Lambda)$ is proper, which in turn holds if and only if there exists a pure-injective indecomposable left Λ-module that is not finitely generated.

There is no known example of a left pure-semisimple ring that is not of finite representation type. Such a ring R would have to be left artinian and there would be a finitely generated indecomposable left R-module ${}_RM$ such that $-\otimes_R \bar{M} \notin \mathcal{O}(\text{fin-}({}_R\mathcal{C}))$.

An object $X \in \mathcal{C}$ is *uniserial* if the lattice of subobjects of X is totally ordered by the subobject relation. For an object X to be uniserial it suffices that the finitely generated subobjects be totally ordered. Hence $A \in \text{coh-}\mathscr{C}$ is uniserial if and only if the lattice $L_{\text{coh-}\mathscr{C}}(A)$ is a total order. A subquotient of a uniserial object is obviously also uniserial. Let $\text{uni-}\mathscr{C} \subseteq \text{coh-}\mathscr{C}$ denote the smallest Serre subcategory of coh-$\mathscr{C}$ to contain every uniserial object of coh-$\mathscr{C}$. By Proposition 3.1,

$$\text{uni-}\mathscr{C} = \bigvee\{A \in \text{coh-}\mathscr{C}:\ A \text{ uniserial}\},$$

and it follows that $A \in \text{uni-}\mathscr{C}$ if and only if there is a finite filtration of A,

$$A = A_0 \geqslant A_1 \geqslant \ldots \geqslant A_n = 0,$$

by coherent subobjects of A such that every factor A_i/A_{i+1} is uniserial. In particular, we have that $\text{uni-}\mathscr{C} \supseteq \text{fin-}\mathscr{C}$.

Proposition 7.10 [**27**, Theorem 10.2]. *Let $E \in \mathscr{C}$ be an injective object. If* $\text{uni-}(\mathscr{C}(E)) \neq 0$, *then E has an indecomposable coproduct factor.*

Proof. Localizing at $\mathscr{S}(E)$, we may assume that $\mathscr{C} = \mathscr{C}(E)$. Let $A \in \text{uni-}\mathscr{C}$ be non-zero. Then $\text{Hom}_{\mathscr{C}}(A, E) \neq 0$, so there is a non-zero morphism $\eta\colon A \to E$. The image $X = \text{Im}\,\eta$ is a uniserial subobject of E. The injective envelope $E(X)$ is therefore an indecomposable factor of E.

An object $W \in \mathscr{C}$ is called *distributive* if the lattice of subobjects of W satisfies the distributive law

$$X \cap (Y + Z) = X \cap Y + X \cap Z$$

for all subobjects X, Y and Z of W. To check that W is distributive it suffices to verify the distributive law for the finitely generated subobjects of W. Thus a coherent object $A \in \text{coh-}\mathscr{C}$ is distributive if and only if the lattice $L_{\text{coh-}\mathscr{C}}(A)$ is distributive. Denote by dis-$\mathscr{C}$ the smallest subcategory of coh-$\mathscr{C}$ to contain all distributive objects of coh-$\mathscr{C}$. Clearly $\text{dis-}\mathscr{C} \supseteq \text{uni-}\mathscr{C}$.

Example. A ring R is called *serial* if it contains a set $\{e_i\}_{i=1}^n$ of primitive idempotent elements such that ${}_RR = \bigoplus_{i=1}^n {}_RRe_i$ (and hence $R_R = \bigoplus_{i=1}^n e_iR_R$) and each of the projective R-modules ${}_RRe_i$ (e_iR_R) is a uniserial left (right) R-module. It is proved in [**10, 29**] that each of the functors $(e_iR, -) \in \text{coh-}({}_R\mathscr{C})$ is distributive. Hence $(R, -) \in \text{dis-}({}_R\mathscr{C})$ and therefore $\text{dis-}({}_R\mathscr{C}) = \text{coh-}({}_R\mathscr{C})$.

Suppose that W is a coherent object of $\mathscr{C}$ that is not distributive. Then there must be three coherent subobjects A, B and C of W for which the inequality

$$(A \cap B) + (A \cap C) \leqslant A \cap (B + C)$$

is strict. In other words, the basic open subset $\mathscr{O}(A \cap (B + C)/(A \cap B) + (A \cap C))$ of the Ziegler spectrum is non-empty. Thus arises the open subset

$$\mathscr{O}_{\text{dis}}(W) := \bigcup_{A,B,C \leqslant W} \mathscr{O}(A \cap (B + C)/(A \cap B) + (A \cap C))$$

with the property that $\mathscr{O}_{\text{dis}}(W) \neq \varnothing$ if and only if W is not a distributive object. Localizing at the corresponding Serre subcategory $\mathscr{S}$ of coh-$\mathscr{C}$ gives a distributive object $W_{\mathscr{S}} \in \mathscr{C}/\vec{\mathscr{S}}$.

By [**7**, Proposition IV.1.6], an object $A \in \text{coh-}\mathscr{C}$ is distributive if and only if A has no coherent subquotient isomorphic to $B \coprod B$ for some $B \in \text{coh-}\mathscr{C}$. A subquotient of a distributive coherent object is therefore also distributive.

Proposition 7.11 [**10**, Proposition 2.4]. *Suppose that the trivial Serre subcategory* $0 \subseteq \text{coh-}\mathscr{C}$ *is* $\cap$*-irreducible. Then every distributive object* $A \in \text{coh-}\mathscr{C}$ *is uniserial.*

Proof. Let B and C be coherent subobjects of the distributive object A. The subquotient $B + C/(B \cap C) \cong B/(B \cap C) \coprod C/(B \cap C)$ is then distributive. As A is distributive, $B/(B \cap C)$ and $C/(B \cap C)$ have no common subquotient and therefore $\sqrt{}(B/(B \cap C)) \cap \sqrt{}(C/(B \cap C)) = 0$. By hypothesis one of $B/(B \cap C)$ and $C/(B \cap C)$ is zero and therefore $B \leqslant C$ or $C \leqslant B$.

The next local-global relation generalizes an observation of C. U. Jensen [**18**] for commutative rings.

Theorem 7.12. *The following are equivalent for* $A \in \text{coh-}\mathscr{C}$.

(1) *The object* A *is distributive.*

(2) *For every Serre subcategory* $\mathscr{S} \subseteq \text{coh-}\mathscr{C}$, $A_{\mathscr{S}}$ *is* $\mathscr{C}/\vec{\mathscr{S}}$*-distributive.*

(3) *For every* $\cap$*-irreducible Serre subcategory* $\mathscr{S} \subseteq \text{coh-}\mathscr{C}$, $A_{\mathscr{S}}$ *is* $\mathscr{C}/\vec{\mathscr{S}}$*-uniserial.*

(4) *For every* $E \in \text{Zg}(\mathscr{C})$, $A_{\mathscr{S}(E)}$ *is* $\mathscr{C}(E)$*-uniserial.*

(5) *For every* $E \in \max(\mathscr{C})$, $A_{\mathscr{S}(E)}$ *is* $\mathscr{C}(E)$*-uniserial.*

Proof. Condition (2) follows from (1) by Proposition 6.2. By Proposition 7.11, Condition (3) is a special case of Condition (2). Condition (4) is a special case of Condition (3). To see that (5) implies (1), suppose that A is not distributive. Then the open subset $\mathscr{O}_{\text{dis}}(A)$ is non-empty and so meets the dense subset $\max(\mathscr{C})$. But if $E \in \mathscr{O}_{\text{dis}}(A)$, then $A_{\mathscr{S}(E)}$ is not $\mathscr{C}(E)$-distributive and hence not $\mathscr{C}(E)$-uniserial.

Given $W \in \text{coh-}\mathscr{C}$, it is clear that $E \in \mathscr{O}_{\text{dis}}(W)$ if and only if $W_{\mathscr{S}(E)}$ is not $\mathscr{C}(E)$-uniserial. It follows from the previous theorem that

$$\mathscr{O}_{\text{dis}}(W) = \bigcup_{A,B \leqslant W} \mathscr{O}(A/A \cap B) \cap \mathscr{O}(A + B/A).$$

Example. A *valuation* ring is a commutative ring R that is uniserial as an R-module. Such a ring is certainly serial. For certain valuation rings R, Puninsky [**30**] and Salce [**32**] have proved the existence of pure-injective R-modules E without indecomposable summands. Thus $\text{dis-}({}_R\mathscr{C}(E)) = \text{coh-}({}_R\mathscr{C}(E))$, while $\text{uni-}({}_R\mathscr{C}(E)) = 0$.

8. *The Grothendieck group*

In this section, we consider the Grothendieck group $K_0(\text{coh-}\mathscr{C})$ of the category of coherent objects of $\mathscr{C}$. Theorem 3.8 is then applied to study the characters of Crawley-Boevey [**8**].

Let $K_0(\text{coh-}\mathscr{C}, \oplus)$ denote the free abelian group on the isomorphism types $[A]$

of objects A in coh-$\mathscr{C}$, modulo the relations $[A \coprod B] = [A] + [B]$. By [**35**, Theorem 1.10], the equation $[A] = [B]$ holds in $K_0(\text{coh-}\mathscr{C}, \oplus)$ if and only if there is a $C \in \text{coh-}\mathscr{C}$ such that $A \coprod C \cong B \coprod C$. The *Grothendieck group* $K_0(\text{coh-}\mathscr{C})$ of coh-$\mathscr{C}$ is the quotient of $K_0(\text{coh-}\mathscr{C}, \oplus)$ by the relations $[A] - [B] + [C]$ for every short exact sequence $0 \to A \to B \to C \to 0$ in coh-$\mathscr{C}$. The subset $K_0^+(\text{coh-}\mathscr{C}) \subseteq K_0(\text{coh-}\mathscr{C})$ of elements having the form $[A]$ where $A \in \text{coh-}\mathscr{C}$ clearly forms a submonoid. This monoid is the positive cone of a pre-order $\leqslant$ with which $K_0(\text{coh-}\mathscr{C})$ is endowed,

$$K_0^+(\text{coh-}\mathscr{C}) = \{x \in K_0(\text{coh-}\mathscr{C}):\ x \geqslant 0\}.$$

The pre-order $\leqslant$ is a transitive relation satisfying the property that for all x, y and z in $K_0(\text{coh-}\mathscr{C})$, $x \leqslant y$ if and only if $x + z \leqslant y + z$. For more on this, see [**14**, § 15].

The duality D: $(\text{coh-}({}_R\mathscr{C}))^{\text{op}} \to \text{coh-}\mathscr{C}_R$ induces an isomorphism

$$K_0(D):\ K_0(\text{coh-}({}_R\mathscr{C})) \to K_0(\text{coh-}\mathscr{C}_R).$$

If the ring R is commutative, ${}_R\mathscr{C} \cong \mathscr{C}_R$ and so $K_0(D)$ is an automorphism of $K_0(\text{coh-}({}_R\mathscr{C}))$ which, because $D^2 = 1$, is an involution.

Example. If $R = Z$, the ring of integers, then $K_0(D)$ is the identity automorphism. Equivalently, $[A] = [DA]$ for every coherent object A of coh-$({}_R\mathscr{C})$. To verify this, it suffices to prove it for some set of generators, for example the projective objects $A = (M, -)$. As $(Z, -) = - \otimes_Z Z = D(Z, -)$, it holds for $A = (Z, -)$. For the cyclic groups $Z(n)$, one has the exact sequence

$$0 \longrightarrow (Z(n), -) \longrightarrow - \otimes_Z Z \xrightarrow{-\otimes n} - \otimes_Z Z \longrightarrow - \otimes_Z Z(n) \longrightarrow 0$$

which gives the equation $[(Z(n), -)] = [- \otimes Z(n)]$ in the Grothendieck group.

A subgroup V of $K_0(\text{coh-}\mathscr{C})$ is called *convex* [**14**, p. 213] if whenever $a, c \in V$ and $a \leqslant b \leqslant c$, then $b \in V$. If V is a convex subgroup of $K_0(\text{coh-}\mathscr{C})$, then the subcategory

$$\mathscr{S}_V := \{A \in \text{coh-}\mathscr{C}:\ [A] \in V\}$$

of coh-$\mathscr{C}$ is Serre. For, suppose that $0 \to A \to B \to C \to 0$ is a short exact sequence in coh-$\mathscr{C}$. If $A, C \in \mathscr{S}_V$, then $[A], [C] \in V$ and so $[B] = [A] + [C] \in V$. On the other hand, if $[B] \in V$, then because $[A], [C] \leqslant [B]$, they also belong to V. We shall call a Serre subcategory *convex* if it is of the form $\mathscr{S}_V$ for some convex subgroup V of $K_0(\text{coh-}\mathscr{C})$.

A subgroup V of $K_0(\text{coh-}\mathscr{C})$ is called *directed* [**14**, p. 213] if every element of V is a difference of elements of $V \cap K_0^+(\text{coh-}\mathscr{C})$. For example, if $\mathscr{S}$ is a Serre subcategory of coh-$\mathscr{C}$, then the subgroup $[\mathscr{S}]$ of $K_0(\text{coh-}\mathscr{C})$ generated by the elements $[S]$, with $S \in \mathscr{S}$, is directed. If V is a directed subgroup of $K_0(\text{coh-}\mathscr{C})$, its convex hull may be described as

$$\text{Con}(V) := \{x:\ -v \leqslant x \leqslant v \text{ for some } v \in V, v \geqslant 0\}.$$

This is again a directed group because $x + v = x - (-v) \geqslant 0$ and $x = (x + v) - v$.

Proposition 8.1. *The map $\mathscr{S} \mapsto [\mathscr{S}]$ is an inclusion-preserving bijective correspondence between the set of convex Serre subcategories of* coh-$\mathscr{C}$ *and the set of directed convex subgroups of* $K_0(\text{coh-}\mathscr{C})$.

Proof. It is clear that if V is a directed convex subgroup of $K_0(\text{coh-}\mathscr{C})$, then $V = [\mathscr{S}_V]$ and so the map $\mathscr{S} \mapsto [\mathscr{S}]$ is surjective. It remains to show that if $\mathscr{S}$ is a convex Serre subcategory, then $[\mathscr{S}]$ is a convex subgroup. Consider the convex hull $V = \text{Con}([\mathscr{S}])$ of the subgroup $[\mathscr{S}]$. As $\mathscr{S}$ is convex, it must be that $\mathscr{S} = \mathscr{S}_V$. But $[\mathscr{S}_V] = V$.

The proposition may be used to define the convex hull $\text{Con}(\mathscr{S})$ of a Serre subcategory $\mathscr{S}$ as the convex Serre subcategory corresponding to the convex subgroup $\text{Con}([\mathscr{S}])$. It may be characterized as

$$\text{Con}(\mathscr{S}) = \{C \in \text{coh-}\mathscr{C}\colon\ [C] \leqslant [S] \text{ for some } S \in \mathscr{S}\}.$$

Example [**14**, Corollary 15.21]. If R is a von Neumann regular ring that is unit-regular, then every Serre subcategory of coh-$\mathscr{C}$ is convex.

The exactness of the inclusion functor $\mathscr{S} \subseteq \text{coh-}\mathscr{C}$ induces a morphism $K_0(\mathscr{S}) \to K_0(\text{coh-}\mathscr{C})$ of pre-ordered abelian groups (Proposition 6.1) whose image is $[\mathscr{S}]$. The localization functor $(-)_{\mathscr{S}}$ is also exact and so the function $[A] \mapsto [A_{\mathscr{S}}]$ induces a well-defined morphism $K_0(-)_{\mathscr{S}}\colon K_0(\text{coh-}\mathscr{C}) \to K_0(\text{coh-}\mathscr{C}/\vec{\mathscr{S}})$ of pre-ordered abelian groups. This morphism $K_0(-)_{\mathscr{S}}$ is evidently an epimorphism whose kernel contains $[\mathscr{S}]$.

Proposition 8.2 [**35**, Corollary 5.14]. *Let $\mathscr{S}$ be a Serre subcategory of* coh-$\mathscr{C}$. *Then the sequence $K_0(\mathscr{S}) \to K_0(\text{coh-}\mathscr{C}) \to K_0(\text{coh-}\mathscr{C}/\vec{\mathscr{S}}) \to 0$ is exact.*

If $\mathscr{S}$ is a convex subcategory of coh-$\mathscr{C}$, the proposition implies that the trivial Serre subcategory $0 \subseteq \text{coh-}\mathscr{C}/\vec{\mathscr{S}}$ is convex. This is equivalent to the Grothendieck group $K_0(\text{coh-}\mathscr{C}/\vec{\mathscr{S}})$ being partially ordered by $\leqslant$. The proposition also implies that if $[A] = 0$ in $K_0(\text{coh-}\mathscr{C}/\vec{\mathscr{S}})$, then by convexity of $\mathscr{S}$, $A \in \mathscr{S}$.

A morphism $\xi\colon K_0(\text{coh-}\mathscr{C}) \to Z$ of pre-ordered abelian groups (where Z is endowed with the standard partial order) is called a *character* [**8**, § 5]. A necessary and sufficient condition for a group morphism $\mu\colon K_0(\text{coh-}\mathscr{C}) \to Z$ to be a character is that $\mu([A]) \geqslant 0$ for each $A \in \text{coh-}\mathscr{C}$. A finite sum of characters is obviously also a character. To each such character $\xi\colon K_0(\text{coh-}\mathscr{C}) \to Z$, we may associate the Serre subcategory

$$\mathscr{S}(\xi) := \{[A]\colon\ \xi([A]) = 0\}.$$

Proposition 8.3. *Let $\xi\colon K_0(\text{coh-}\mathscr{C}) \to Z$ be a character. The Serre subcategory $\mathscr{S}(\xi) \subseteq$ coh-$\mathscr{C}$ is convex.*

Proof. Let $A \in \text{Con}(\mathscr{S}(\xi))$. There is a $C \in \text{coh-}\mathscr{C}$ such that $\xi([C]) = 0$ and $[A] \leqslant [C]$. Then $\xi([A]) = 0$ and so $A \in \text{Con}(\mathscr{S}(\xi))$.

For example, if $\mathscr{C}$ is a locally finite Grothendieck category, the function $A \mapsto l(A)$, which assigns to a coherent object A its length, induces a character $l\colon K_0(\text{coh-}\mathscr{C}) \to Z$. The proposition then says that the trivial Serre subcategory $0 = \mathscr{S}(l) \subseteq \text{coh-}\mathscr{C}$ is convex.

Proposition 8.4. *Let* ξ: $K_0(\text{coh-}\mathcal{C}) \to Z$ *be a character and let* $\mathcal{S}$ *be the convex Serre subcategory* $\mathcal{S}(\xi)$. *Then* $\mathcal{C}/\vec{\mathcal{S}}$ *is locally finite and* ξ *factors as*

$$\begin{array}{ccc} K_0(\text{coh-}\mathcal{C}) & \xrightarrow{K_0(-)_{\mathcal{S}}} & K_0(\text{coh-}\mathcal{C}/\vec{\mathcal{S}}) \\ {\scriptstyle\xi}\downarrow & \swarrow {\scriptstyle\mu} & \\ Z & & \end{array}$$

where μ: $K_0(\text{coh-}\mathcal{C}/\vec{\mathcal{S}}) \to Z$ *is again a character.*

Proof. By Proposition 8.2, ξ factors through $K_0(\text{coh-}\mathcal{C}/\vec{\mathcal{S}})$, so it remains to show that $\mathcal{C}/\vec{\mathcal{S}}$ is locally finite. We shall prove by induction on $\xi([B])$, where $B \in \text{coh-}\mathcal{C}$, that $l_{\mathcal{C}/\vec{\mathcal{S}}}(B_{\mathcal{S}}) \leqslant \xi([B])$. Note that

$$\xi([B]) = 0 \quad \Leftrightarrow \quad B \in \mathcal{S} \quad \Leftrightarrow \quad l_{\mathcal{C}/\vec{\mathcal{S}}}(B_{\mathcal{S}}) = 0.$$

First we show that if $\xi([B]) = 1$, then $B_{\mathcal{S}}$ must be $\mathcal{C}/\mathcal{S}$-simple. By Proposition 2.14, every $\mathcal{C}/\vec{\mathcal{S}}$-finitely generated subobject $X \leqslant B_{\mathcal{S}}$ is of the form $A_{\mathcal{S}}$ for some finitely generated, hence coherent, subobject $A \leqslant B$. Consider the short exact sequence

$$0 \to A \to B \to C \to 0.$$

It must be that $\xi([A]) = 0$ or that $\xi([C]) = 0$ which means that $A_{\mathcal{S}} = 0$ or that $A_{\mathcal{S}} = B_{\mathcal{S}}$. To prove the induction step, we assume that $B_{\mathcal{S}}$ is not $\mathcal{C}/\vec{\mathcal{S}}$-simple and consider a short exact sequence as above where neither of the coherent objects A or C lies in $\mathcal{S}$. Then $\xi([A])$, $\xi([C]) < \xi([B])$ and so by the induction hypothesis, we get

$$l_{\mathcal{C}/\vec{\mathcal{S}}}(B_{\mathcal{S}}) = l_{\mathcal{C}/\vec{\mathcal{S}}}(A_{\mathcal{S}}) + l_{\mathcal{C}/\vec{\mathcal{S}}}(C_{\mathcal{S}}) \leqslant \xi([A]) + \xi([C]) = \xi([B]).$$

A character is called *irreducible* if it is not the sum of two non-zero characters. It is clear by Proposition 8.2 that in the previous proposition the induced character μ is irreducible if and only if the given character ξ is. If $\mathcal{C}$ is locally finite and $S \in \mathcal{C}$ is simple, a character ξ_S: $K_0(\text{coh-}\mathcal{C}) \to Z$ is obtained by assigning to a coherent object A the number of times the simple object S occurs as a composition factor of A. This character ξ_S is clearly irreducible. For, if $\xi_S = \xi_1 + \xi_2$, then for one of the ξ_i, say ξ_1, we have that $\xi_i([S]) = 1$. Then $\xi_S \leqslant \xi_1$ and hence $\xi_2 = 0$.

Proposition 8.5. *Let* $\mathcal{C}$ *be locally finite and* ξ: $K_0(\text{coh-}\mathcal{C}) \to Z$ *a character on* coh-$\mathcal{C}$. *If* $\{S_i\}_{i \in I}$ *is the set of isomorphism types of the simple objects of* $\mathcal{C}$, *then* ξ *is expressible uniquely as a* (*possibly infinite*) Z-*linear combination of the characters* ξ_{S_i}

$$\xi = \sum_{i \in I} \xi([S_i])\xi_{S_i}.$$

Thus ξ *is irreducible if and only if it is one of the* ξ_{S_i} *for some* $i \in I$.

Proof. Every element $x \in K_0(\text{coh-}\mathcal{C})$ is a Z-linear combination $x = \sum_{i \in I} n_i[S_i]$

of the generators $[S_i]$. Now $\xi(x) = \sum_{i\in I} n_i \xi([S_i])$ where $n_i = \xi_{S_i}(x)$, so the equality holds. To prove uniqueness, suppose $\sum_{i\in I} n_i \xi_{S_i} = 0$. Then for every $j \in I$, $n_j = \sum_{i\in I} n_i \xi_{S_i}[S_j] = 0([S_j]) = 0$.

The previous two propositions give the following result of Crawley-Boevey.

THEOREM 8.6 [**8**, Theorems 5.1, 5.2]. *Every character* ξ: $K_0(\text{coh-}\mathscr{C}) \to Z$ *is uniquely expressible as a (possibly infinite) Z-linear combination*

$$\xi = \sum_{i\in I} n_i \xi_i$$

of irreducible characters ξ_i.

Let R be a ring and ξ: $K_0(\text{coh-}({}_R\mathscr{C})) \to Z$ a character. As every coherent object C of ${}_R\mathscr{C}$ is a subquotient of some $(-\otimes_R R)^n$, it is clear that ξ is non-zero if and only if $\xi([-\otimes_R R]) \neq 0$. It follows that every character ξ: $K_0(\text{coh-}({}_R\mathscr{C})) \to Z$ is a finite sum of irreducible characters.

9. *Endofinite modules*

In this final section we consider irreducible characters as well as the characters on the Grothendieck group of the category coh-$({}_R\mathscr{C})$ of coherent generalised left R-modules. Most of the results are due to Crawley-Boevey [**8**], but the methods of this paper shed new light on them.

EXAMPLE. Let ${}_RM$ be an endofinite R-module with endomorphism ring $T = \text{End}_R\, M$. By Proposition 7.6, the category ${}_R\mathscr{C}(M)$ is locally finite and one may define the character

$$\mu_M([C]) := l_{({}_R\mathscr{C}(M))}(C_{\mathscr{S}(M)}).$$

By Proposition 6.3, $\mu_M([C]) \leq l_T(C, -\otimes_R M)$. On the other hand, Proposition 7.5 implies that $\mu_M([C]) \geq l_T(C, -\otimes_R M)$. Hence we get a more useful description of μ_M in the form of the equality

$$\mu_M([C]) = l_T(C, -\otimes_R M).$$

In particular, $\mu_M([-\otimes_R R]) = l_T(-\otimes_R R, -\otimes_R M) = l_T({}_TM)$ is the endolength of ${}_RM$. Note also that $\mathscr{S}(\mu_M) = \mathscr{S}({}_RM)$.

Let ξ: $K_0(\text{coh-}\mathscr{C}) \to Z$ be an irreducible character and $\mathscr{S}$ the Serre subcategory $\mathscr{S}(\xi)$ of coh-$\mathscr{C}$. The proof of Proposition 8.4 shows that the character ξ dominates the character $l_{\mathscr{C}/\mathscr{S}}(-)_{\mathscr{S}}$. But since ξ is irreducible, these two characters are in fact equal. The category $\mathscr{C}/\mathscr{S}$ is locally finite and since the character induced by the length function is irreducible, the category $\mathscr{C}/\mathscr{S}$ has only one simple object S (up to isomorphism) and the character induced on $K_0(\text{coh-}(\mathscr{C}/\vec{\mathscr{S}}))$ by ξ is ξ_S. The Ziegler spectrum $\text{Zg}(\mathscr{C}/\mathscr{S})$ has only one point, namely $E = E(S)$, which, by Proposition 3.6, may be thought of as a closed point of $\text{Zg}(\mathscr{C})$. By Theorem 3.8, the category $\mathscr{S}(\xi)$ is a maximal Serre subcategory of coh-$\mathscr{C}$.

We call a closed point $E \in \mathrm{Zg}(\mathscr{C})$ *endofinite* if the localization $\mathscr{C}/\vec{\mathscr{S}}(E)$ is locally finite. If $T = \mathrm{End}_{\mathscr{C}} E$, an argument as in the example shows that ξ is equal to the character μ_E: $K_0(\text{coh-}\mathscr{C}) \to Z$ defined by

$$\mu_E([C]) := l_T(C, E).$$

THEOREM 9.1. *There is a bijective correspondence between the endofinite closed points E of* $\mathrm{Zg}(\mathscr{C})$ *and the irreducible characters ξ: $K_0(\text{coh-}\mathscr{C}) \to Z$. The correspondence is given by the map*

$$E \mapsto \mu_E = l_T(-, E)$$

where $T = \mathrm{End}_{\mathscr{C}} E$.

Proof. We have just proved that the map is surjective. It is also one-to-one, because if E and F are distinct endofinite closed points of $\mathrm{Zg}(\mathscr{C})$, there is a coherent object $C \in \mathscr{C}$ such that $E \in \mathscr{O}(C)$ while $F \notin \mathscr{O}(C)$. The two characters μ_E and μ_F are then distinct since $\mu_F([C]) = 0 < \mu_E([C])$.

COROLLARY 9.2. *Let R be a ring. There is a bijective correspondence between the endofinite indecomposable left R-modules M and the irreducible characters ξ: $K_0(\text{coh-}_R\mathscr{C}) \to Z$. The correspondence is given by the map*

$$M \mapsto \mu_M = l_T(-, - \otimes_R M)$$

where $T = \mathrm{End}_R M$.

For the category of generalized R-modules, the characters on the Grothendieck group may be thought of as the Sylvester rank functions of Schofield [**33**, Chapter 7]. To see this, let $K_0(\text{mod-}R, \oplus)$ be the abelian group presented by the generators $[M]$ where $M \in \text{mod-}R$ and the relations $[M \oplus N] - [M] - [N]$. By Yoneda's Lemma, the representation functor $M_R \mapsto (M_R, -)$ is an equivalence between the category mod-R and the subcategory of projective objects of coh-$({}_R\mathscr{C})$. As every short exact sequence of projective objects is split-exact, $K_0(\text{mod-}R, \oplus)$ is isomorphic to the Grothendieck group of the projective objects. Recall that every object $C \in \text{coh-}({}_R\mathscr{C})$ has a projective resolution

$$0 \longrightarrow (K_R, -) \longrightarrow (N_R, -) \xrightarrow{(f,-)} (M_R, -) \longrightarrow C \longrightarrow 0.$$

By [**35**, Theorem 4.4], the map χ: $K_0(\text{coh-}({}_R\mathscr{C})) \to K_0(\text{mod-}R, \oplus)$ given by

$$C \mapsto [M] - [N] + [K]$$

is a well-defined isomorphism of abelian groups.

Thus we may associate to each character ξ: $K_0(\text{coh-}({}_R\mathscr{C})) \to Z$ the rank function ρ: mod-$R \to Z$ defined by $\rho(M) = \xi\chi^{-1}([M])$. These are precisely the rank functions ρ satisfying the following two conditions.

(1) We have $\rho(M \oplus N) = \rho(M) + \rho(N)$. This just asserts that the induced map $\rho\chi = \xi$: $K_0(\text{coh}-{}_R\mathscr{C}) \to Z$ is a morphism of abelian groups.

(2) If f: $M_R \to N_R$ is a morphism in mod-R, then $\rho(M) - \rho(N) + \rho(\mathrm{Coker}\, f) \geqslant 0$. This condition asserts that for a $C \in \text{coh-}({}_R\mathscr{C})$ with a projective resolution as above, $\rho\chi([C]) = \rho(M) - \rho(N) + \rho(\mathrm{Coker}\, f) \geqslant 0$.

EXAMPLE. Let Λ be an artin algebra. Then every finitely generated indecomposable Λ-module ${}_{\Lambda}M$ is endofinite and so gives rise to an irreducible character μ_M. The intersection V of the kernels of these characters is a convex subgroup of $K_0(\text{coh-}({}_{\Lambda}\mathscr{C}))$ such that $\mathscr{S}_V = 0$. Thus the trivial Serre subcategory 0 of coh-$({}_{\Lambda}\mathscr{C})$ is convex.

Let $C \in \text{coh-}\mathscr{C}$ and consider a finite filtration $\mathscr{F}$,

$$C = C_0 \geqslant C_1 \geqslant \ldots \geqslant C_{n+1} = 0$$

of C by coherent subobjects. Let $\mathscr{O}(\mathscr{F}) = \bigcap_{i \leqslant n} \mathscr{O}(C_i/C_{i+1})$. This is a finite intersection of basic open subsets and is therefore open. If $E \in \mathscr{O}(\mathscr{F})$, then all of the inclusions in the localized filtration $\mathscr{F}_{\mathscr{S}(E)}$,

$$C_{\mathscr{S}(E)} = (C_0)_{\mathscr{S}(E)} \geqslant (C_1)_{\mathscr{S}(E)} \geqslant \ldots \geqslant (C_{n+1})_{\mathscr{S}(E)} = 0,$$

are proper and therefore $l_{\mathscr{C}(E)}(C_{\mathscr{S}(E)}) > n$ (we allow, of course, for the possibility that the length may be infinite).

DEFINITION. For each natural number n, define $\mathscr{O}_n(C) := \bigcup(\mathscr{F})$ where the index set runs over all filtrations of C of length $n+1$.

Note that the open subset $\mathscr{O}_0(C)$ is just the basic open subset $\mathscr{O}(C)$.

THEOREM 9.3. *Let $C \in \text{coh-}\mathscr{C}$. Then $\mathscr{O}_n(C) = \{E \in \text{Zg}(\mathscr{C})\colon\ l_{\mathscr{C}(E)}(C_{\mathscr{S}(E)}) > n\}$.*

Proof. We have already noted that if $E \in \mathscr{O}_n(C)$, then $l_{\mathscr{C}(E)}(C_{\mathscr{S}(E)}) > n$. Conversely, if $l_{\mathscr{C}(E)}(C_{\mathscr{S}(E)}) > n$, then there is a filtration $\mathscr{G}$ of length $n+1$ of $C_{\mathscr{S}(E)}$ by $\mathscr{C}(E)$-coherent subobjects. Proposition 2.14 shows that $\mathscr{G}$ is the localization $\mathscr{G} = \mathscr{F}_{\mathscr{S}(E)}$ of some filtration $\mathscr{F}$ of length $n+1$ of C by $\mathscr{C}$-coherent subobjects. But then $E \in \mathscr{O}(\mathscr{F}) \subseteq \mathscr{O}_n(C)$.

COROLLARY 9.4. *Let μ_E: $K_0(\text{coh-}\mathscr{C}) \to Z$ be an irreducible character. For every $C \in \text{coh-}\mathscr{C}$, $E \in \mathscr{O}_n(C)$ if and only if $\xi_E([C]) > n$.*

DEFINITION. *Let R be a ring. Define $\text{Zg}_n({}_R\mathscr{C}) = \text{Zg}({}_R\mathscr{C}) \setminus \mathscr{O}_n(- \otimes_R R)$.*

The subset $\text{Zg}_n({}_R\mathscr{C})$ is a closed and hence quasi-compact subset of the left Ziegler spectrum of R. A pure-injective indecomposable module ${}_RM$ belongs to this set if and only if it has endolength at most n. Since all of these points are closed, the subspace $\text{Zg}_n({}_R\mathscr{C})$ satisfies the separation axiom T_1. The subsets $\text{Zg}_n({}_R\mathscr{C})$ and $\text{Zg}_n(\mathscr{C}_R)$ are homeomorphic via the map D defined as follows. If $M \in \text{Zg}_n({}_R\mathscr{C})$, then μ_M: $K_0(\text{coh-}({}_R\mathscr{C})) \to Z$ is an irreducible character such that $\mu_M([- \otimes_R R]) \leqslant n$. The character $\mu_M K_0(D)$: $K_0(\text{coh-}\mathscr{C}_R) \to Z$ is then also irreducible and so has the form μ_N for some endofinite indecomposable right R module N_R. Furthermore, $N \in \text{Zg}_n(\mathscr{C}_R)$ since $\mu_N([R \otimes_R -]) \leqslant n$. Define $DM = N$.

Let us describe $\text{Zg}_1({}_R\mathscr{C})$, the subspace of all *endosimple* left R-modules, that is, of those modules ${}_RM$ that are simple as modules over their endomorphism ring. If $\Delta = \text{End}_R M$ is the local endomorphism ring, then because M is a faithful simple Δ-module, Δ must be a (not necessarily commutative) field and ${}_{\Delta}M$ a one-dimensional vector space over Δ. The action of R gives a ring homomorphism α: $R \to \Delta^{\text{op}} = \text{End}_{\Delta} \Delta$. As ${}_RM$ is indecomposable, Δ^{op} contains no proper field containing the image of α. Conversely, every such ring homomorphism $\alpha\colon R \to \Delta^{\mathrm{op}}$ gives rise to an endosimple indecomposable left R-module ${}_R\Delta$.

The *field spectrum* of R in the sense of Cohn [**7**, p. 410] is a topological space whose points are the epic R-fields, that is, ring homomorphisms $\alpha\colon R \to \Delta^{\mathrm{op}}$ (up to isomorphism) with the property that Δ^{op} is a minimal field containing the image of α. It is easy to see that if

$$\begin{array}{ccc} R & \longrightarrow & \Delta_1^{\mathrm{op}} \\ \downarrow & \swarrow_{f} & \\ \Delta_2^{\mathrm{op}} & & \end{array}$$

is an isomorphism of epic R-fields, then $f^{\mathrm{op}}\colon {}_R\Delta_1 \to {}_R\Delta_2$ is an isomorphism of R-modules. Conversely, the epic property may be used to prove that an isomorphism $g\colon {}_R\Delta_1 \to {}_R\Delta_2$ of endosimple indecomposable R-modules induces an isomorphism of epic R-fields. The topology on the field spectrum of R is given by the following basis of open subsets. To each square $n \times n$ matrix A, the basic open subset

$$\mathcal{O}(A) := \{\Delta^{\mathrm{op}}\colon\ A \in M_n(\Delta^{\mathrm{op}}) \text{ is invertible}\}$$

is associated. That $\Delta^{\mathrm{op}} \in \mathcal{O}(A)$ means the morphism $-\otimes A\colon -\otimes_R R^n \to -\otimes_R R^n$ in coh-$({}_R\mathscr{C})$ becomes invertible upon localisation at $\mathscr{S}(\Delta)$. Note that in $\mathrm{Zg}_1({}_R\mathscr{C})$, the subset $\mathcal{O}(A)$ corresponds to the open subset $\mathcal{O}_{n-1}((-\otimes_R R^n)/\mathrm{Ker}(-\otimes A))$. If R is a commutative ring, the field spectrum of R is homeomorphic to the Zariski spectrum.

Define the *constructible field spectrum* of R to be the topological space whose points are those of the field spectrum and with a basis of open subsets given by finite boolean combinations of the subsets $\mathcal{O}(A)$. In $\mathrm{Zg}_1({}_R\mathscr{C})$, the complement of the open subset $\mathcal{O}(A)$ corresponds to the open subset $\mathcal{O}(\mathrm{Ker}(-\otimes A))$, so the map $\Delta \mapsto \Delta^{\mathrm{op}}$ is a continuous bijection from $\mathrm{Zg}_1({}_R\mathscr{C})$ to the constructible field spectrum. As the space $\mathrm{Zg}_1({}_R\mathscr{C})$ is quasi-compact, so is the constructible field spectrum of R. Because the field spectrum of R satisfies the separation axiom T_0 [**7**, Ex. 13, p. 412], the constructible field spectrum is Hausdorff. As the bijection above is continuous, the subspace $\mathrm{Zg}_1({}_R\mathscr{C})$ must also be Hausdorff. The following is now immediate.

THEOREM 9.5. *The subspace* $\mathrm{Zg}_1({}_R\mathscr{C})$ *is homeomorphic via the map* $\Delta \mapsto \Delta^{\mathrm{op}}$ *to the constructible field spectrum of R.*

Suppose that Λ is an artin algebra and let c denote the length of Λ as a module over its centre. If ${}_\Lambda M$ is a Λ-module of length at most m, then its length over the centre, and so its endolength, are bounded by mc. If there were an infinite family of indecomposable Λ-modules of length m, then there would be infinitely many finitely generated indecomposable Λ-modules of endolength at most mc.

THEOREM 9.6 (Crawley-Boevey) [**8**, Theorem 9.6]. *Let* Λ *be an artin algebra. If* *there are infinitely many finitely generated indecomposable* Λ*-modules of endolength at most n, then there is an indecomposable* Λ*-module G of endolength at most n that is not finitely generated.*

Proof. The hypothesis asserts that $\mathrm{Zg}_n({}_\Lambda\mathscr{C})$ contains infinitely many isolated points. As $\mathrm{Zg}_n({}_\Lambda\mathscr{C})$ is quasi-compact, this set of isolated points has an accumulation point $E \in \mathrm{Zg}_n({}_\Lambda\mathscr{C})$. But then $E = - \otimes_R G$ where ${}_RG$ is an endofinite indecomposable Λ-module of endolength at most n that is not finitely generated.

An endofinite indecomposable Λ-module that is not finitely generated is called *generic.* A generic module is a Λ-module that corresponds to an irreducible character μ_G, but does not belong to the open subset $\mathscr{O}(\text{fin-}({}_\Lambda\mathscr{C}))$ of isolated points. Crawley-Boevey [**8**] has proved that for an infinite artin algebra Λ, the existence of a generic Λ-module implies the existence of an infinite family of finitely generated indecomposable Λ-modules of some fixed length. Thus the second Brauer–Thrall Conjecture may be rephrased as follows.

The Second Brauer–Thrall Conjecture. If Λ is an artin algebra that is not of finite representation type, then there exists a generic Λ-module.

In the quest for a generic Λ-module, one may argue as follows. If Λ is not of finite representation type, then fin-$\mathscr{C}$ is a proper Serre subcategory of coh-$\mathscr{C}$. By Zorn's Lemma, there is a maximal proper Serre subcategory fin-$\mathscr{C} \subseteq \mathscr{S} \subset$ coh-$\mathscr{C}$. The Ziegler spectrum of $\mathscr{C}/\vec{\mathscr{S}}$ is then an indiscrete topological space. The problem is to find suitable conditions on $\mathscr{S}$ which imply that it has the form $\mathscr{S}(\xi)$ for some irreducible character ξ. The following generalises a test due to Ziegler.

Proposition 9.7 [**37**, Lemma 8.11]. *Let* $\mathscr{C}$ *be a locally coherent Grothendieck category such that* $\mathrm{Zg}(\mathscr{C})$ *is indiscrete. If there are an* $A \in$ coh-$\mathscr{C}$ *and* $E \in \max(\mathscr{C})$ *such that E is a coproduct factor of the injective envelope* $E_{\mathscr{C}}(A)$, *then* $\mathscr{C}$ *is locally finite.*

Proof. As $E = E(S)$ for some simple object $S \in \mathscr{C}$ and E is a coproduct factor of an essential extension of A, it must be that S is a subobject of A and is therefore coherent. Now $\sqrt{S}$ clearly consists of those objects of finite length all of whose composition factors are isomorphic to S. By Theorem 3.8, $\sqrt{S} =$ coh-$\mathscr{C}$.

If there exists a maximal Serre subcategory fin-$({}_\Lambda\mathscr{C}) \subseteq \mathscr{S} \subset$ coh-$({}_\Lambda\mathscr{C})$ such that $\mathscr{S} = \mathscr{S}(\xi)$ for some character ξ, then $\mathscr{S}$ is necessarily a convex Serre subcategory. So it makes sense to consider a maximal convex Serre subcategory fin-$({}_\Lambda\mathscr{C}) \subseteq \mathscr{S} \subset$ coh-$({}_\Lambda\mathscr{C})$ and ask whether it corresponds to some irreducible character. The existence of such a maximal convex Serre subcategory is derived from Zorn's Lemma together with the following.

Proposition 9.8. *If* Λ *is an artin algebra, then* fin-$({}_\Lambda\mathscr{C})$ *is a convex Serre subcategory of* coh-$({}_\Lambda\mathscr{C})$.

Proof. First note that $C \in$ fin-$({}_\Lambda\mathscr{C})$ if and only if there are only finitely many finitely generated indecomposable Λ-modules ${}_{\Lambda}M$ such that $\mu_M([C]) \neq 0$. If A belongs to the convex hull Con(fin-$({}_{\Lambda}\mathscr{C})$), then there is a $C \in$ fin-$({}_{\Lambda}\mathscr{C})$ such that $[A] \leqslant [C]$. But then $\mu_M([A]) \neq 0$ for only finitely many finitely generated indecomposables ${}_{\Lambda}M$. Thus fin-$({}_{\Lambda}\mathscr{C})$ = Con(fin-$({}_{\Lambda}\mathscr{C})$) is convex.

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