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The Endomorphism Ring of a Localized Coherent Functor

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Let C be a commutative artinian ring and Λ an artin C-algebra. The category of coherent additive functors A: mod-$\Lambda \to$ Ab on the finitely presented right Λ-modules will be denoted by $\mathrm{Ab}(\Lambda)$. This category is equivalent to the free abelian category over the ring Λ. If $\mathscr{S}_0 \subseteq \mathrm{Ab}(\Lambda)$ is the Serre subcategory of the finite length objects of $\mathrm{Ab}(\Lambda)$ and $A \in \mathrm{Ab}(\Lambda)$, it is proved that the endomorphism ring $\mathrm{End}_{\mathrm{Ab}(\Lambda)/\mathscr{S}_0} A_{\mathscr{S}_0}$ of the localized object $A_{\mathscr{S}_0}$ is a locally artin C-algebra. This is used to show that the Krull-Gabriel dimension of the category $\mathrm{Ab}(\Lambda)$ cannot equal 1. In particular, this holds for finite rings.

Let C be a commutative artinian ring and Λ an artin C-algebra. The category [1] of coherent additive functors A: mod-$\Lambda \to$ Ab on the finitely presented right Λ-modules is denoted by $\mathrm{Ab}(\Lambda)$. This category is equivalent [6] to the free abelian category over the ring Λ. The point of this paper is to prove the following theorem.

THEOREM 3.2. *Let Λ be an artin C-algebra and $\mathscr{S}_0$ the Serre subcategory of* $\mathrm{Ab}(\Lambda)$ *of finite length objects. If $A \in$* $\mathrm{Ab}(\Lambda)$, *then the endomorphism ring* $\mathrm{End}_{\mathrm{Ab}(\Lambda)/\mathscr{S}_0} A_{\mathscr{S}_0}$ *of the localized object $A_{\mathscr{S}_0}$ is a locally artin C-algebra.*

By a *locally artin* C-algebra we mean a C-algebra every finitely generated subalgebra of which is artin. The proof uses a theorem of Krause [10] that relates restricting $\mathrm{Ab}(\Lambda)$ to a covariantly finite subcategory of mod-Λ with some localization of $\mathrm{Ab}(\Lambda)$.

Taking $A = \mathrm{Hom}_\Lambda(\Lambda, -)$ to be the forgetful functor, the theorem shows that, in the language of Prest [13], the definable scalars of the elementary theory of right Λ-modules without finitely generated direct summands is a locally artin C-algebra.

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Taking A such that $S = A_{\mathscr{S}_0}$ is a simple object of $\mathrm{Ab}(\Lambda)/\mathscr{S}_0$, the theorem shows that $\mathrm{End}_{\mathrm{Ab}(\Lambda)/\mathscr{S}_0} S$ is a division ring that is algebraic over the field $k_S = C/\mathrm{ann}_C(S)$. This case is reminiscent of (and the general theorem inspired by) the result [12, Lemma 0.5] of Buechler and Hrushovski that the division ring of isogenies of a weakly minimal abelian structure with few types is a locally finite field.

Recall from [7] that the Krull−Gabriel dimension KG-dim(Ab(Λ)) of the category Ab(Λ) is equal to zero if $\mathrm{Ab}(\Lambda) = \mathscr{S}_0$, in which case Λ is of finite representation type. The Krull−Gabriel dimension KG-dim(Ab(Λ)) is equal to 1 if $\mathrm{Ab}(\Lambda) \neq \mathscr{S}_0$, but every object of $\mathrm{Ab}(\Lambda)/\mathscr{S}_0$ has finite length. As a consequence of the theorem one has the following.

THEOREM 3.6. *If Λ is an artin C-algebra, then* $\text{KG-dim}(\mathrm{Ab}(\Lambda)) \neq 1$.

This has been shown by Krause [11] to hold when Λ is a finite-dimensional algebra over an algebraically closed field. The proof of Theorem 3.6 follows from a result of Crawley-Boevey [5] which asserts that whenever an artin C-algebra Λ has a generic left module (a module is called *generic* if it is indecomposable, is endofinite, and is not finitely generated), then it has a special generic left module ${}_\Lambda G$ with the property that the division ring $\mathrm{top}(\mathrm{End}\,{}_\Lambda G)$ contains an element transcendental over the subfield $k_G = C/\mathrm{ann}_C(\mathrm{top}(\mathrm{End}\,{}_\Lambda G))$.

Throughout the paper, R will denote an associative ring with unit $1 \in R$. The category of finitely presented right R-modules is denoted by mod-R.

1. COVARIANTLY FINITE SUBCATEGORIES

A full additive subcategory $\mathscr{C} \subseteq$ mod-R (which includes the condition that $\mathscr{C}$ is closed under isomorphism and direct summands) is *covariantly finite* (in mod-R) if, given $M \in$ mod-R, there is a morphism $a_M\colon M \to M_{\mathscr{C}}$ with $M_{\mathscr{C}} \in \mathscr{C}$ such that any morphism $f\colon M \to N$ with $N \in \mathscr{C}$ factors through a_M. This means that there is a $\mathscr{C}$-morphism $f_{\mathscr{C}}\colon M_{\mathscr{C}} \to N$ so that the diagram

$$\begin{array}{ccc} M & \xrightarrow{a_M} & M_{\mathrm{C}} \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle f_{\mathscr{C}}} & \\ N & & \end{array}$$

commutes. The morphism $a_M\colon M \to M_{\mathscr{C}}$ is called a left *$\mathscr{C}$-approximation* of M. Covariantly finite subcategories were introduced by Auslander and Smalø [4]. Throughout this paper, $\mathscr{C}$ will denote a covariantly finite subcategory of mod-R.

The category $\mathscr{C}$ has pseudo-cokernels [2]. This means that every $\mathscr{C}$-morphism $f: M \to N$ has a *pseudo-cokernel*, a morphism $g: N \to K$ in $\mathscr{C}$ such that $gf = 0$ and any $\mathscr{C}$-morphism $h: N \to L$ for which $hf = 0$ will factor through g. To find a pseudo-cokernel in $\mathscr{C}$ of a $\mathscr{C}$-morphism $f: M \to N$, take the cokernel $g_0: N \to K$ in the category mod-R. Then the composition $a_K g_0: N \to K_{\mathscr{C}}$ is a pseudo-cokernel of f in $\mathscr{C}$.

Let $(\mathscr{C}, \text{Ab})$ denote the category of additive functors $F: \mathscr{C} \to \text{Ab}$, with morphisms the natural transformations between functors. By Yoneda's lemma [14, Prop. IV.7.3], an object in $(\mathscr{C}, \text{Ab})$ is a finitely generated projective object if and only if it is *representable*, that is, if it is isomorphic to one of the functors

$$(M, -) := \text{Hom}_{\mathscr{C}}(M, -) = \text{Hom}_R(M, -)|_{\mathscr{C}},$$

where $M \in \mathscr{C}$. Yoneda's lemma also yields that every $(\mathscr{C}, \text{Ab})$-morphism $\eta: (N, -) \to (M, -)$ between representable functors has the form $\eta = (f, -)$, where $f: M \to N$ is a $\mathscr{C}$-morphism.

Let $\alpha: F \to G$ be a morphism in $(\mathscr{C}, \text{Ab})$. The *kernel* and *image* of α are the subfunctors $\text{Ker}\,\alpha$ and $\text{Im}\,\alpha$ of F and G, respectively, defined by

$$(\text{Ker}\,\alpha)(M) := \text{Ker}\,\alpha(M) \quad \text{and} \quad (\text{Im}\,\alpha)(M) := \text{Im}\,\alpha(M).$$

It is easily verified that the morphism α is a monomorphism (respectively, epimorphism) if and only if $\text{Ker}\,\alpha = 0$ (respectively, $\text{Im}\,\alpha = G$). A sequence

$$F \xrightarrow{\alpha} G \xrightarrow{\beta} H$$

of objects in $(\mathscr{C}, \text{Ab})$ is *exact* if $\text{Im}\,\alpha = \text{Ker}\,\beta$. This is equivalent to the condition that, for every object $M \in \mathscr{C}$, the sequence

$$F(M) \xrightarrow{\alpha(M)} G(M) \xrightarrow{\beta(M)} H(M)$$

of abelian groups is exact. In particular, the restriction functor $(-)|_{\mathscr{C}}$: $(\text{mod-}R, \text{Ab}) \to (\mathscr{C}, \text{Ab})$ defined by $F \mapsto F|_{\mathscr{C}}$ is exact.

A functor $F \in (\mathscr{C}, \text{Ab})$ is *finitely generated* if there is an epimorphism η: $(M, -) \to F$ from a representable functor to F; it is *finitely presented* if there is an exact sequence

$$(N, -) \xrightarrow{(f, -)} (M, -) \to F \to 0 \tag{1}$$

in $(\mathscr{C}, \text{Ab})$, where $f: M \to N$ is a $\mathscr{C}$-morphism. This exact sequence is called a *projective presentation* of F. A finitely presented functor F is *coherent* [1] if every finitely generated subfunctor is also finitely presented. Auslander proved [2, Prop., p. 102] that the coherent functors form an *exact* subcategory coh-$(\mathscr{C}, \mathrm{Ab})$ of $(\mathscr{C}, \mathrm{Ab})$. This means that coh-$(\mathscr{C}, \mathrm{Ab})$ is an abelian category and that the inclusion functor coh-$(\mathscr{C}, \mathrm{Ab}) \subseteq (\mathscr{C}, \mathrm{Ab})$ is exact.

Using the hypothesis that the category $\mathscr{C}$ has pseudo-cokernels, Auslander showed [2, Prop., p. 102] that every finitely presented functor $F \in (\mathscr{C}, \mathrm{Ab})$ is coherent. To see this, it is enough by the exactness of the subcategory coh-$(\mathscr{C}, \mathrm{Ab})$, to verify that every representable functor $(M, -)$ is coherent. For then both of the projective objects $(M, -)$ and $(N, -)$ appearing in the projective presentation (1) of F are coherent and, therefore, so must be $(M, -)/\mathrm{Im}(f, -) = F$. Now every finitely generated subfunctor G of $(M, -)$ is the image of some morphism $(g, -)$: $(K, -) \to (M, -)$, where g: $M \to K$ is a morphism in $\mathscr{C}$. Let h: $K \to L$ be a pseudo-cokernel of g in $\mathscr{C}$. It follows from the definition of pseudo-cokernel that the sequence

$$(L, -) \xrightarrow{(h, -)} (K, -) \xrightarrow{(g, -)} (M, -)$$

is exact and gives a projective presentation of $\mathrm{Im}(g, -) = G$.

For brevity, we shall denote by $\mathrm{Ab}(R)$ the abelian subcategory of coherent or, equivalently, finitely presented objects of (mod-R, Ab). This notation is justified by the result of Freyd [6] that $\mathrm{Ab}(R)$ is the free abelian category over R. We shall now note how the restriction functor $(-)|_{\mathscr{C}}$: $\mathrm{Ab}(R) \to (\mathscr{C}, \mathrm{Ab})$ preserves coherent objects. Because this functor is exact, it is enough to verify that the restriction to $\mathscr{C}$ of a representable functor $(M, -)$, $M \in$ mod-R, is a coherent object of $(\mathscr{C}, \mathrm{Ab})$. For then the coherent functor F with projective presentation (1) will take on the value of the cokernel in $(\mathscr{C}, \mathrm{Ab})$ of $(f, -)|_{\mathscr{C}}$: $(N, -)|_{\mathscr{C}} \to (M, -)|_{\mathscr{C}}$, a coherent functor. Let a_M: $M \to M_{\mathscr{C}}$ be a left $\mathscr{C}$-approximation of M and let g: $M_{\mathscr{C}} \to N$ be the cokernel of a_M in mod-R. If a_N: $N \to N_{\mathscr{C}}$ is a left $\mathscr{C}$-approximation of N, one can easily verify that the sequence

$$(N_{\mathscr{C}}, -) \xrightarrow{(a_N g, -)} (M_{\mathscr{C}}, -) \xrightarrow{(a_M, -)|_{\mathscr{C}}} (M, -)|_{\mathscr{C}} \to 0$$

of $(\mathscr{C}, \mathrm{Ab})$-objects is exact and therefore gives a projective presentation of $(M, -)|_{\mathscr{C}}$.

The restriction functor $(-)|_{\mathscr{C}}$: $\mathrm{Ab}(R) \to$ coh-$(\mathscr{C}, \mathrm{Ab})$ is dense and full. This is so because every $(\mathscr{C}, \mathrm{Ab})$-morphism α: $F \to G$ between finitely presented objects is induced by a morphism of the respective projective presentations

$$\begin{array}{ccccccc} (N, -) & \xrightarrow{(f,-)} & (M, -) & \longrightarrow & F & \longrightarrow & 0 \\ {\scriptstyle (h,-)}\downarrow & & {\scriptstyle (k,-)}\downarrow & & \downarrow{\scriptstyle \alpha} & & \\ (N', -) & \xrightarrow[(g,-)]{} & (M', -) & \longrightarrow & G & \longrightarrow & 0. \end{array}$$

Then $hg = fk$, so if $\alpha'\colon F' \to G'$ is the $\mathrm{Ab}(R)$-morphism induced by the same morphism of projective presentations, then $F'|_{\mathscr{C}} = F$, $G'|_{\mathscr{C}} = G$, and $\alpha'|_{\mathscr{C}} = \alpha$.

2. LOCALIZATION AT A SERRE SUBCATEGORY

A full subcategory $\mathscr{S}$ of $\mathrm{Ab}(R)$ is called *Serre* if, for every short exact sequence

$$0 \to F \to G \to H \to 0$$

in $\mathrm{Ab}(R)$, G is an object of $\mathscr{S}$ if and only if F and H are objects of $\mathscr{S}$. If $\mathscr{X}$ is a subcategory of mod-R, then the subcategory

$$\mathscr{S}(\mathscr{X}) := \{A \in \mathrm{Ab}(R)\colon A(\mathscr{X}) = 0\}$$

of $\mathrm{Ab}(R)$ is Serre. It is clear that an intersection of Serre subcategories of $\mathrm{Ab}(R)$ is again Serre. A Serre subcategory $\mathscr{S}$ of $\mathrm{Ab}(R)$ is *finitely generated* if there are finitely many objects $S_1, S_2, \dots, S_n$ in $\mathscr{S}$ such that $\mathscr{S}$ is the smallest Serre subcategory that contains each of the S_i.

Given a Serre subcategory $\mathscr{S}$ of $\mathrm{Ab}(R)$, define as in [7, Ch. III] the abelian category $\mathrm{Ab}(R)/\mathscr{S}$, the *localization* of $\mathrm{Ab}(R)$ at $\mathscr{S}$, whose objects are the objects A of $\mathrm{Ab}(R)$, but denoted with a subscript $A_{\mathscr{S}}$, and whose morphisms are given as follows. For $A, B \in \mathrm{Ab}(R)$,

$$\mathrm{Hom}_{\mathrm{Ab}(R)/\mathscr{S}}(A_{\mathscr{S}}, B_{\mathscr{S}}) = \varinjlim \mathrm{Hom}_{\mathrm{Ab}(R)}(A', B/B'),$$

where $A' \leq A$ is such that $A/A' \in \mathscr{S}$ and $B' \leq B$ is such that $B' \in \mathscr{S}$. The set of such pairs is a partially ordered directed set with respect to the relation $(A', B/B') \preccurlyeq (A'', B/B'')$, which holds if and only if $A'' \leq A'$ and $B' \leq B''$. The direct limit is indexed by this partial order.

The functor $(-)_{\mathscr{S}}\colon \mathrm{Ab}(R) \to \mathrm{Ab}(R)/\mathscr{S}$ has the universal property that whenever $T\colon \mathrm{Ab}(R) \to \mathscr{A}$ is an exact functor to an abelian category $\mathscr{A}$, such that $T(\mathscr{S}) = 0$, then T factors through $(-)_{\mathscr{S}}$ uniquely, that is, there is a unique exact functor $T/\mathscr{S}\colon \mathrm{Ab}(R)/\mathscr{S} \to \mathscr{A}$ such that the diagram

$$\begin{array}{ccc} \mathrm{Ab}(R) & \xrightarrow{(-)_{\mathscr{S}}} & \mathrm{Ab}(R)/\mathscr{S} \\ {\scriptstyle T}\downarrow & \swarrow {\scriptstyle T/\mathscr{S}} & \\ \mathscr{A} & & \end{array}$$

commutes. As an example of this, let $\mathscr{S} \subseteq \mathscr{S}'$ be an inclusion of Serre subcategories. An exact functor $\mathrm{Ab}(R)/\mathscr{S} \to \mathrm{Ab}(R)/\mathscr{S}'$ defined by $A_{\mathscr{S}} \mapsto A_{\mathscr{S}'}$ is then induced. The verification of the next proposition is straightforward.

PROPOSITION 2.1. *Let $\mathscr{S}$ be a Serre subcategory of* $\mathrm{Ab}(R)$ *which is a directed union $\mathscr{S} = \bigcup_i \mathscr{S}_i$ of Serre subcategories $\mathscr{S}_i$ of* $\mathrm{Ab}(R)$. *Then, for $A, B \in \mathrm{Ab}(R)$,*

$$\mathrm{Hom}_{\mathrm{Ab}(R)/\mathscr{S}}(A_{\mathscr{S}}, B_{\mathscr{S}}) = \varinjlim \mathrm{Hom}_{\mathrm{Ab}(R)/\mathscr{S}_i}(A_{\mathscr{S}_i}, B_{\mathscr{S}_i}),$$

where the structural morphisms of the direct limit are induced by the inclusions of the $\mathscr{S}_i$.

The special case of the proposition that will interest us is when the $\mathscr{S}_i$ are taken to be the finitely generated Serre subcategories of $\mathscr{S}$.

Let $\mathscr{C} \subseteq \text{mod-}R$ be a covariantly finite subcategory. Then the exact functor $(-)|_{\mathscr{C}}\colon \mathrm{Ab}(R) \to \text{coh-}(\mathscr{C}, \mathrm{Ab})$ induces an exact functor

$$\Phi_{\mathscr{C}} := (-)|_{\mathscr{C}}/\mathscr{S}(\mathscr{C})\colon \mathrm{Ab}(R)/\mathscr{S}(\mathscr{C}) \to \text{coh-}(\mathscr{C}, \mathrm{Ab}).$$

As $(-)|_{\mathscr{C}}$ is dense and full, so is $\Phi_{\mathscr{C}}$. In the sequel, we shall need the following theorem of Krause.

THEOREM 2.2 [10, Thm. 3.4]. *Let $\mathscr{C} \subseteq \text{mod-}R$ be a covariantly finite subcategory. The functor $\Phi_{\mathscr{C}}$ above, defined by $A_{\mathscr{S}(\mathscr{C})} \mapsto A|_{\mathscr{C}}$, is an equivalence of categories.*

3. ARTIN ALGEBRAS

Let C be a commutative artinian ring. A C-algebra Λ is *artin* if the C-module ${}_C\Lambda$ is finitely generated. If Λ is an artin C-algebra and $M \in \text{mod-}\Lambda$, then the endomorphism ring $\mathrm{End}_{\Lambda} M$ is also an artin C-algebra. To see this, consider an epimorphism $\Lambda^n \to M$ and apply the left exact functor $(-, M)$ to get an embedding of C-modules $\mathrm{End}_\Lambda M \to (\Lambda^n, M) = M^n$.

Similarly, it follows that if $\mathscr{C} \subseteq$ mod-Λ is a covariantly finite subcategory and $A \in$ coh-$(\mathscr{C}, \mathrm{Ab})$, then the endomorphism ring $\mathrm{End}_{(\mathscr{C}, \mathrm{Ab})} A$ is an artin C-algebra. For, consider an epimorphism $(M, -) \to A$. Evaluated at M_Λ, this gives an epimorphism $\mathrm{End}_\Lambda M_\Lambda \to A(M)$ of C-modules. Applying now the left exact functor $\mathrm{Hom}_{(\mathscr{C}, \mathrm{Ab})}(?, A)$ to the first epimorphism gives, by Yoneda's lemma, an embedding $\mathrm{End}_{(\mathscr{C}, \mathrm{Ab})} A \to A(M)$ of C-modules. Thus, $\mathrm{End}_{(\mathscr{C}, \mathrm{Ab})} A$ is a finitely generated C-module. We document this as follows.

PROPOSITION 3.1. *Let Λ be an artin C-algebra. If $\mathscr{C} \subseteq$ mod-Λ is a covariantly finite subcategory and $A \in$ coh-$(\mathscr{C}, \mathrm{Ab})$, then the endomorphism ring $\mathrm{End}_{(\mathscr{C}, \mathrm{Ab})} A$ is an artin C-algebra.*

The simple objects S of $\mathrm{Ab}(\Lambda)$ are in bijective correspondence with the finitely generated indecomposable right Λ-modules. This correspondence is given by the rule $S \mapsto M_\Lambda$, where $(M, -) \to S$ is a projective cover of S in $\mathrm{Ab}(\Lambda)$. If we denote this simple object by S_M, then a finitely generated Λ-module X_Λ lies in the support of S_M if and only if X_Λ has a direct summand isomorphic to M_Λ.

Call a C-algebra Γ *locally artin* if every finitely generated C-subalgebra of Γ is artin. It is clear that a C-algebra is locally artin if and only if it is a direct limit of artin C-algebras.

THEOREM 3.2. *Let Λ be an artin C-algebra and $\mathscr{S}_0$ the Serre subcategory of $\mathrm{Ab}(\Lambda)$ consisting of the objects of finite length. If $A \in \mathrm{Ab}(\Lambda)$, then the endomorphism ring $\mathrm{End}_{\mathrm{Ab}(\Lambda)/\mathscr{S}_0} A_{\mathscr{S}_0}$ is a locally artin C-algebra.*

Proof. By Proposition 2.1, it is enough to prove that, for every finitely generated $\mathscr{S}' \subseteq \mathscr{S}$, the endomorphism ring $\mathrm{End}_{\mathrm{Ab}(\Lambda)/\mathscr{S}'} A_{\mathscr{S}'}$ of the object A localized at $\mathscr{S}'$ is an artin C-algebra. We shall show how $\mathscr{S}' = \mathscr{S}(\mathscr{C})$ for some covariantly finite subcategory $\mathscr{C} \subseteq$ mod-Λ. By Theorem 2.2, the endomorphism ring $\mathrm{End}_{\mathrm{Ab}(\Lambda)/\mathscr{S}'} A_{\mathscr{S}'}$ is isomorphic to the C-algebra $\mathrm{End}_{(\mathscr{C}, \mathrm{Ab})} A|_{\mathscr{C}}$, which, by Proposition 3.1, is an artin C-algebra.

Let $\mathscr{S}' \subseteq \mathscr{S}_0$ be finitely generated. Then there are finitely many simple objects $S_1, S_2, \ldots, S_n$ that generate $\mathscr{S}'$. Thus every object of $\mathscr{S}'$ has a composition series with factors among the S_i. Let $M_1, M_2, \ldots, M_n$ be the finitely presented indecomposables corresponding to the S_i and $X \in$ mod-Λ. Then $F(X) = 0$ for every $F \in \mathscr{S}'$ if and only if X does not have a direct summand isomorphic to one of the M_i. Let

$$\mathscr{C} := \{X \in \text{mod-}\Lambda\colon F(X) = 0 \text{ for every } F \in \mathscr{S}\}.$$

Auslander and Smalø (cf. [3, Prop. 2.9]) have shown this category to be covariantly finite in mod-Λ. Furthermore, $\mathscr{S}' = \mathscr{S}(\mathscr{C})$ since any $F \in \mathscr{S}(\mathscr{C})$ must have finite length with factors among the S_i. ∎

Let us consider some special cases of this result. The contravariant functor $M_\Lambda \mapsto (M_\Lambda, -)$ from mod-Λ to Ab(Λ) is a full and left exact embedding. Thus $\mathrm{End}_{\mathrm{Ab}(\Lambda)}(M, -) \cong (\mathrm{End}_\Lambda M)^{\mathrm{op}}$ and so one may think of the object $(M, -)_{\mathscr{S}_0}$ as representing the "localization" of M_Λ at $\mathscr{S}_0$.

COROLLARY 3.3. *Let $M_\Lambda \in$ mod-Λ. Localization of the functor $(M_\Lambda, -)$ at $\mathscr{S}_0$ gives a homomorphism of C-algebras*

$$\lambda_M : (\mathrm{End}_\Lambda M)^{\mathrm{op}} \to \mathrm{End}_{\mathrm{Ab}(\Lambda)/\mathscr{S}_0}(M, -)_{\mathscr{S}_0},$$

where the domain is artin and the codomain locally artin.

When $M_\Lambda = \Lambda_\Lambda$, this gives a C-algebra homomorphism λ_Λ: $\Lambda^{\mathrm{op}} \to \mathrm{End}_{\mathrm{Ab}(\Lambda)/\mathscr{S}_0}(\Lambda, -)_{\mathscr{S}_0}$.

Let $S \in \mathrm{Ab}(\Lambda)/\mathscr{S}_0$ be simple. Then $k_S = C/(C \cap \mathrm{ann}_C(S))$ is a field and the theorem takes on the following form.

COROLLARY 3.4. *Let $S \in \mathrm{Ab}(\Lambda)/\mathscr{S}_0$ be simple. Then the division ring $\Delta_S = \mathrm{End}_{\mathrm{Ab}(\Lambda)/\mathscr{S}_0} S$ is algebraic over k_S.*

Let us explain how the simple objects of (mod-Λ, Ab)$/\mathscr{S}_0$ correspond to certain pure-injective indecomposable left Λ-modules. The category (mod-Λ, Ab) is a locally coherent Grothendieck category whose injective objects [9, Appendix B] are precisely those isomorphic to functors of the form $- \otimes_\Lambda M$ where ${}_\Lambda M$ is a pure-injective left Λ-module. If $\vec{\mathscr{S}}_0 \subseteq$ (mod-Λ, Ab) is the Serre subcategory of objects F isomorphic to a direct limit of objects from $\mathscr{S}_0$, then [8, Thm. 2.6]

$$\text{coh-}\left[(\text{mod-}\Lambda, \mathrm{Ab})/\vec{\mathscr{S}}_0\right] \cong \mathrm{Ab}(\Lambda)/\mathscr{S}_0.$$

Furthermore, the subcategory $\vec{\mathscr{S}}_0$ is localizing [7, Ch. III.3, Cor. 1] meaning that the localization functor from (mod-Λ, Ab) to (mod-Λ, Ab)$/\vec{\mathscr{S}}_0$ has a right adjoint that allows us to identify (mod-Λ, Ab)$/\vec{\mathscr{S}}_0$ with a full subcategory of (mod-Λ, Ab) in such a way [7, Ch. III.3, Cor. 2] that the injective objects of (mod-Λ, Ab)$/\vec{\mathscr{S}}_0$ are those that are injective as objects of the category (mod-Λ, Ab).

If $S \in \mathrm{Ab}(\Lambda)/\mathscr{S}_0$ is simple, the injective envelope $E(S)$ of S in the category (mod-Λ, Ab)$/\vec{\mathscr{S}}_0$ is of the form $- \otimes_\Lambda U[S]$, where ${}_\Lambda U[S]$ is a pure-injective, necessarily indecomposable, left Λ-module. The endomorphism ring

$$\operatorname{End}_\Lambda U[S] = \operatorname{End}_{(\text{mod-}\Lambda,\,\text{Ab})}(-\otimes_\Lambda U[S]) = \operatorname{End}_{(\text{mod-}\Lambda,\,\text{Ab})} E(S)$$

is a local ring whose residue division ring, or *top*, is just

$$\operatorname{top}(\operatorname{End}_\Lambda U[S]) = \operatorname{top}\big(\operatorname{End}_{(\text{mod-}\Lambda,\,\text{Ab})} E(S)\big) = \operatorname{End}_{(\text{mod-}\Lambda,\,\text{Ab})/\vec{\mathscr{S}}_0} S = \Delta_S.$$

The division ring $\operatorname{top}(\operatorname{End}_\Lambda U[S]) = \Delta_S$ is thus, by Corollary 3.4, algebraic over the field $k_S = C/C \cap \operatorname{ann}_C[\operatorname{top}(\operatorname{End}_\Lambda U[S])]$.

A left Λ-module ${}_\Lambda M$ is said to be *endofinite* if it has finite length when considered as a module over the endomorphism ring $\operatorname{End}_\Lambda M$ in the natural way. Every finitely generated module over an artin C-algebra has finite length as a C-module and is therefore endofinite. Every endofinite module ${}_\Lambda M$ is pure-injective [5, Thm. 3.1 and Prop. 4.1]. An indecomposable endofinite module ${}_\Lambda G$ is called *generic* [5, Sect. 7] if it is not finitely generated.

Recall [7, Ch. IV.1] that the Krull–Gabriel dimension KG-dim(Ab(Λ)) of the category Ab(Λ) is equal to zero if $\text{Ab}(\Lambda) = \mathscr{S}_0$. In this case Λ is of *finite representation type*, that is, there are, up to isomorphism, only finitely many finitely generated indecomposable right Λ-modules. The Krull–Gabriel dimension KG-dim(Ab(Λ)) is equal to 1 if $\text{Ab}(\Lambda) \neq \mathscr{S}_0$, but every object of $\text{Ab}(\Lambda)/\mathscr{S}_0$ has finite length.

PROPOSITION 3.5. *If* KG-dim(Ab(Λ)) = 1, *then the simple objects of* $\text{Ab}(\Lambda)/\mathscr{S}_0$ *correspond to the generic left* Λ*-modules*.

Proof. Suppose KG-dim(Ab(Λ)) = 1. Then the category $(\text{mod-}\Lambda, \text{Ab})/\vec{\mathscr{S}}_0$ is locally finite and therefore every injective object is of the form $-\otimes_\Lambda M$ with ${}_\Lambda M$ endofinite [8, Prop. 7.6]. Because it is a locally finite category, the function $S \mapsto E(S)$ that associates to a simple object its injective envelope is not only one-to-one, but, as every object has a simple subobject, it is a bijective correspondence between the simple objects and the injective indecomposable objects of $(\text{mod-}\Lambda, \text{Ab})/\vec{\mathscr{S}}_0$.

The indecomposable injective objects of $(\text{mod-}\Lambda, \text{Ab})/\vec{\mathscr{S}}_0$ are precisely those indecomposable injective objects $-\otimes_\Lambda U$ of $(\text{mod-}\Lambda, \text{Ab})$ that do not have a subobject from $\mathscr{S}_0$ or, equivalently, that do not have a simple subobject. But an indecomposable object of the form $-\otimes_\Lambda U$ has a simple subobject [11, Lemma 11.2] if and only if ${}_\Lambda U$ is finitely generated. Thus the simple objects of $(\text{mod-}\Lambda, \text{Ab})/\vec{\mathscr{S}}_0$ correspond to the pure-injective indecomposables ${}_\Lambda U$ that are not finitely generated. Because $(\text{mod-}\Lambda, \text{Ab})/\vec{\mathscr{S}}_0$ is locally finite, these are precisely the generics. ∎

THEOREM 3.6. *If Λ is an artin C-algebra, then* KG-dim(Ab(Λ)) $\neq 1$.

Proof. Let us assume that KG-dim(Ab(Λ)) = 1 and aim for a contradiction. By Proposition 3.5, there exists a generic left Λ-module and, furthermore, every generic ${}_{\Lambda}G$ has the property that the top(End$_{\Lambda}$ G) is algebraic over the field $k_G = C/\mathrm{ann}_C(\mathrm{top}(\mathrm{End}_{\Lambda} G))$. But this contradicts a theorem of Crawley-Boevey [5, Thm. 9.6], which asserts that if a generic left Λ-module exists, then there is one such module ${}_{\Lambda}G$ such that the top(End$_{\Lambda}$ G) contains an element transcendental over the field k_G. ∎

For the case of a finite-dimensional algebra over an algebraically closed field, Theorem 3.6 has been proved by Krause [11, Cor. 11.4]. Because every finite ring R is an artin $Z(R)$-algebra, where $Z(R)$ denotes the center of R, the following corollary is a special case of the theorem.

COROLLARY 3.7. *If R is a finite ring, then* KG-dim(Ab(R)) $\neq 1$.

The following example of a ring R for which KG-dim(Ab(R)) = 1 was shown to me by Goodearl. Let k be a field and denote by k^{ω} the ring of countable sequences with entries in k, with addition and multiplication defined componentwise. Let R be the subring of k^{ω} that consists of eventually constant sequences. The ring R is commutative and von Neumann regular so that the category Ab(R) is equivalent, via the functor $(M, -) \mapsto M$, to the category mod-R of finitely presented R-modules. The simple objects of mod-R are the ideals $e_i R$, where e_i denotes an element of R with a unique nonzero entry that occurs in the ith place. Localizing at the Serre subcategory of finitely presented R-modules of finite length, one obtains a category equivalent to the category of finitely presented modules over the ring $R/k^{(\omega)}$, where $k^{(\omega)} = \sum_{n<\omega} e_N R$ is the maximal ideal of R of sequences that are eventually zero. As $R/k^{(\omega)}$ is a field, this is the category of its finite-dimensional vector spaces and so every object has finite length.

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