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A test for finite representation type

Ivo Herzog [1]

Department of Mathematics, Brandeis University, Waltham, MA 02254, USA

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Abstract

Let R be a left pure-semisimple ring. It is proved that if R has self-duality or if R is a polynomial identity ring, then R is of finite representation type. If there exists an example of a left pure-semisimple ring which is not of finite representation type, we show that then there exists an example which is hereditary, but not right artinian. We reduce the question of the existence of such an example to a problem regarding simple bimodules over division rings.

Introduction

A ring R (associative, with identity) is called *left pure-semisimple* if every left (unital) R-module is a direct sum of indecomposable left R-modules. Chase [8] showed that every such ring is left artinian. A left pure-semisimple ring is said to be of *finite representation type* if there are (up to isomorphism) only finitely many indecomposable left R-modules. There is no known example of a left pure-semisimple ring which is not of finite representation type and it is conjectured that such an example does not exist:

The Pure-semisimple Conjecture. Every left pure-semisimple ring is of finite representation type.

Warfield [22] verified the Pure-semisimple Conjecture for commutative rings and Auslander [4] extended this result to artin algebras. Some of the history of this problem is chronicled in [10, p. xiii] and [28].

[1] Supported by a National Science Foundation Postdoctoral Fellowship. Current address: Mathematisches Institut, University of Freiburg, Albertstr. 236, 79104 Freiburg, Germany. Email: ivo@sun1.mathematik.uni-freiburg.de.

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The left pure-semisimple rings have been characterized model-theoretically [12] as those rings R such that every complete theory of left R-modules is totally transcendental. The model-theoretic import of the Pure-semisimple Conjecture is then derived from the fact [17, Section 11.4] that a ring R is of finite representation type iff every complete theory of left R-modules has finite Morley rank. And although the results of this article are cloaked in the language of algebra and categories, many proofs rely heavily on model-theoretic techniques [17,25]. For example, our main application of Yoneda's Lemma is a homogeneity principle we call the Generator Lemma.

The main body of this paper rests upon an analysis (Section 4) of the right *preinjective* modules and left *preprojective* modules over a left pure-semisimple ring. A systematic study of such modules was undertaken by Auslander and Smalø [6] in the context of artin algebras. The left preinjective modules over a left pure-semisimple ring have been studied by Zimmermann-Huisgen [27]. We prove, in the fourth section, that if R is a left pure-semisimple ring, then there are only finitely many preprojective left R-modules and that this number is a bound on the number of preinjective right R-modules. The test for finite representation type is then stated as follows:

Theorem 5.2. *A left pure-semisimple ring R is of finite representation type iff the number of preprojective left R-modules is equal to the number of preinjective right R-modules.*

From this test it follows that the Pure-semisimple Conjecture holds for rings with self-duality, a class of rings which contains the artin algebras.

Corollary 5.3. *If R is a left pure-semisimple ring and there exists a Morita duality* $\mathcal{D}$: mod-$R \to R$-mod, *then R is of finite representation type.*

If there exists a counterexample to the Pure-semisimple Conjecture, we use the test to obtain a counterexample R which is not right Morita (a ring is *right Morita* if it is right artinian and every indecomposable injective right R-module is finitely generated).

A counterexample R to the Pure-semisimple Conjecture is called *minimal* if given any nontrivial two-sided ideal I of R the quotient ring R/I is of finite representation type. It is obvious that given a counterexample to the conjecture one obtains a minimal counterexample by modding out by a sufficiently large ideal. We prove that a minimal counterexample contains a unique minimal ideal (Proposition 5.8).

In the last section, we generalize certain of Simson's results [21] in the hereditary case. Namely, we apply Auslander's theory of Grassmanians [2] to a minimal counterexample which is not right Morita to get the following theorem:

Theorem 6.5. *If there is a counterexample to the Pure-semisimple Conjecture, then there are division rings F and G and a simple G–F-bimodule ${}_GB_F$ such that the formal lower triangular matrix ring*

$$R_B = \begin{pmatrix} F & 0 \\ B & G \end{pmatrix}$$

is a counterexample to the Pure-semisimple Conjecture which is not right Morita.

In essence, this paper describes a procedure which, given an arbitrary counterexample to the Pure-semisimple Conjecture, produces the counterexample of Theorem 6.5. Keeping track of the operations that are performed to arrive at this last counterexample yields the following theorem:

Theorem 6.7. *Every left pure-semisimple polynomial identity ring is of finite representation type.*

Finally, we prove that in the case the counterexample R_B of Theorem 6.5 is right artinian then the formal lower triangular matrix ring $R_{(B,F)}$ is a counterexample which fails to be right artinian.

Theorem 6.9. *If a counterexample exists to the Pure-semisimple Conjecture, then there is a counterexample R which is hereditary and not right artinian.*

By a ring R we mean an associative ring with identity and by an R-module we mean a unital R-module. The Jacobson radical of R will be denoted by $J(R)$. The category R-Mod is the category of left R-modules while Mod-R is the category of right R-modules, with morphisms always acting on the left. The full subcategory consisting of the finitely presented right R-modules is denoted by mod-R, its sinister analogue by R-mod. The category of abelian groups is denoted by Ab.

The first three sections are of a preliminary nature. In the first, we consider the finitely generated objects of the category (mod-R, Ab) of functors from the category mod-R to the category Ab. We relate these functors to the positive-primitive formulae in the language for right R-modules which, in turn, are intimately related to finite matrix subgroups. In the second section, we describe those rings R such that every finitely generated object of (mod-R, Ab) has a projective cover and we explain the correspondence which exists for such rings between the indecomposable finitely presented right R-modules and the simple objects of (mod-R, Ab). In the third section, we review Auslander's characterizations [3,4] of left pure-semisimple rings in terms of the category (mod-R, Ab).

1. Finitely generated functors

Let R be a ring. A module $M \in \text{Mod-}R$ is *finitely presented* if it is the cokernel of a morphism g between finitely generated free modules. This may be represented by an exact sequence

$$R^m \xrightarrow{g} R^n \to M \to 0$$

which is called a *free presentation* of M. The full subcategory of Mod-R of finitely presented right R-modules is denoted by mod-R. If $M, N \in \text{mod-}R$ and $f : M \to N$ is a morphism, then the cokernel of f is also finitely presented. While the category Mod-R is abelian, the same is not always true of its subcategory mod-R. All of these considerations pertain equally as well to left R-modules, the category of finitely presented left R-modules denoted by R-mod.

The class of covariant additive functors $F : \text{mod-}R \to \text{Ab}$ may be endowed with the structure of a category $(\text{mod-}R, \text{Ab})$ with morphisms the natural transformations of functors. If $\eta : F \to G$ is a morphism in $(\text{mod-}R, \text{Ab})$, then $\operatorname{Ker} \eta$, the *kernel* of η, is defined as $(\operatorname{Ker} \eta)(M) = \operatorname{Ker} \eta_M$ and the *image* of η, $\operatorname{Im} \eta$ is defined as $(\operatorname{Im} \eta)(M) = \eta_M(M)$. These are again objects in $(\text{mod-}R, \text{Ab})$ as is the *cokernel* $\operatorname{Coker} \eta$, defined by $(\operatorname{Coker} \eta)(M) = G(M)/\operatorname{Im} \eta_M$. The natural transformation η is a *monomorphism* (an *epimorphism*) if $\operatorname{Ker} \eta$ ($\operatorname{Coker} \eta$) is zero. It is an *isomorphism* in the categorical sense if it is both a monomorphism and an epimorphism. With these definitions, it is routine to verify that the category $(\text{mod-}R, \text{Ab})$ is abelian.

The objects of $(\text{mod-}R, \text{Ab})$ which first spring to mind are the functors $(M,-) = \operatorname{Hom}_R(M,-)$ where $M \in \text{Mod-}R$. A functor $F \in (\text{mod-}R, \text{Ab})$ is called *representable* if it is isomorphic to such a functor with $M \in \text{mod-}R$. The key result regarding the category $(\text{mod-}R, \text{Ab})$ is the following:

Yoneda's Lemma. *Let $F \in \mathcal{C} = (\text{mod-}R, \text{Ab})$ and $M \in \text{mod-}R$. The morphism of abelian groups*

$$\Theta_{F,M} : \operatorname{Hom}_{\mathcal{C}}((M,-), F) \to F(M)$$

defined by $\Theta_{F,M}(\eta) = \eta_M(1_M)$ is an isomorphism, natural both in F and in M. $\square$

Every morphism $f : M \to N$ in mod-R gives rise to a natural transformation $(f,-) : (N,-) \to (M,-)$. Yoneda's Lemma implies that every natural transformation $\eta : (N,-) \to (M,-)$ with $M, N \in \text{mod-}R$ is of this form.

If $F \in (\text{mod-}R, \text{Ab})$, we say that G is a *subfunctor* of F, or $G \subseteq F$, if for each $M \in \text{mod-}R$, we have the inclusion $G(M) \subseteq F(M)$ of abelian groups and for every morphism $f : M \to N$ in mod-R the Ab-morphism $G(f)$ is the restriction $F(f)|_{G(M)}$. For example, if $\eta : F \to G$ is a natural transformation, then $\operatorname{Ker}\eta$ is a subfunctor of F while $\operatorname{Im}\eta$ is a subfunctor of G. If $G \subseteq F$ is a subfunctor, then the *quotient* functor F/G is defined by $(F/G)(M) = F(M)/G(M)$; the cokernel of a morphism is an example. Note that the subfunctors of F form a complete lattice under the operations of intersection and sum: If $G_1, G_2 \subseteq F$, define $(G_1 \cap G_2)(M) = G_1(M) \cap G_2(M)$ and $(G_1 + G_2)(M) = G_1(M) + G_2(M)$. A functor F is called *simple* if it contains no nontrivial subfunctors.

The functor F is said to be *finitely generated* if there are a module $M \in$ mod-R and $a \in F(M)$ with the property that whenever $G \subseteq F$ and $a \in G(M)$, then $G = F$. Such an element $a \in F(M)$ is called a *generator* of F. For example, the element $1_M \in (M, M)$ is a generator of $(M, -)$. Given $G \in (\text{mod-}R, \text{Ab})$ we say that $a \in G(M)$ generates the functor F if $F \subseteq G$ and $a \in F(M)$ is a generator of F. This functor F may be described in terms of a as $F = \bigcap\{F' \subseteq G \mid a \in F'(M)\}$. It is easily verified that if $a \in F(M)$ is a generator and $\eta : F \to G$ is a morphism in $(\text{mod-}R, \text{Ab})$, then $\eta_M(a) \in G(M)$ is a generator of $\operatorname{Im}\eta$.

Proposition 1.1. *A functor $F \in (\text{mod-}R, \text{Ab})$ is finitely generated iff there is an epimorphism $\eta : (M, -) \to F$ from a representable object to F.*

Proof. Let $\eta : (M, -) \to F$ be a morphism which corresponds, via Yoneda's Lemma, to the element $a \in F(M)$; thus η is determined by $\eta_M(1_M) = a$. Now $\operatorname{Im}\eta$ is the subfunctor of F generated by $a \in F(M)$. Hence $a \in F(M)$ is a generator iff η is an epimorphism. □

The following homogeneity principle will find repeated application in the sequel. It relates the notion of generator to a method for constructing morphisms.

The Generator Lemma. *Let $F \in (\text{mod-}R, \text{Ab})$ be finitely generated with generator $a \in F(M)$. If $c \in F(N)$, $N \in$ mod-R, then there is a morphism $f : M \to N$ such that $F(f)(a) = c$. Conversely, any element $a' \in F(M')$ with this property is a generator of F.*

Proof. For $a \in F(M)$, define the functor Tr_a by

$$\operatorname{Tr}_a(N) = F(M, N)a = \{F(f)(a) \mid f : M \to N\}.$$

This is a subfunctor of F and clearly $a \in \operatorname{Tr}_a(M)$. As $a \in F(M)$ is a generator, it follows that $\operatorname{Tr}_a = F$ and hence that $c \in \operatorname{Tr}_a(N)$.

For the converse, let $a' \in F(M')$ have the stated property and suppose that $G \subseteq F$ with $a' \in G(M')$. We will show that $F \subseteq G$. Indeed, if $c \in F(N)$, then, by hypothesis, there is a morphism $f : M' \to N$ such that $F(f)(a') = c$. But then $G(f)(a') = F(f)(a') = c$ and so $c \in G(N)$. □

Next, we will describe the finitely generated subfunctors of $(R_R, -)$ in model-theoretic terms (cf. [17, Section 2.2]). This will clarify the relationship of these functors to linear algebra over the ring R. Let $\mathcal{L}_R = \langle +, -, 0, r \rangle_{r \in R}$ be the language for right R-modules; the first three symbols are intended to interpret the underlying abelian group structure of a right R-module M, while for each $r \in R$, there is a unary function symbol r (acting on the right) intended to interpret the action of $r \in R$. It is easy to express in the language $\mathcal{L}_R$ the axioms for a right R-module—we denote this set by T_R. Since quantification over the ring R is not permitted in the language $\mathcal{L}_R$, the set of axioms T_R does not in general form a finite set.

A *positive-primitive formula* in the language of right R-modules (abbreviation: right pp-formula over R) is nothing more that an existentially quantified system of linear equations with coefficients in R (acting on the right). More precisely, let A be a row vector with entries in R, say of length n, and let B be a matrix over R having the same number of columns as A; so B is an $m \times n$ matrix. A right pp-formula $\varphi(x)$ over R in the free variable x is a formula of $\mathcal{L}_R$ of the form

$$\exists \mathbf{y}(x, \mathbf{y}) \begin{pmatrix} A \\ B \end{pmatrix} \doteq \mathbf{0} \tag{1}$$

where $\mathbf{y} = (y_1, \ldots, y_m)$ is the m-tuple of the existentially bound variables. Thus a right pp-formula over R is completely determined by a row vector A and a matrix B with the same number of columns. We use the symbol $\rightleftharpoons$ to express equality of pp-formulae. Generally, we will use the Greek letters $\varphi(x)$ or $\psi(x)$ to talk about pp-formulae.

Given a right R-module M, the right pp-formula $\varphi(x)$ above defines the subgroup

$$\varphi(M) = \left\{ a \in M \;\middle|\; \exists c_1, \ldots, c_m \in M \;\; (a, c_1, \ldots, c_m) \begin{pmatrix} A \\ B \end{pmatrix} = \mathbf{0} \right\}$$

of the underlying abelian group structure of M. The relation $a \in \varphi(M)$ is expressed model-theoretically using the satisfaction symbol $M_R \models \varphi(a)$; we say that $a \in M$ is a *realization* of φ. The subgroups of M arising in this fashion are called the *finite matrix subgroups* of M; they were introduced by Gruson and Jensen and, independently, by Zimmermann, as the subgroups of M_R of *finite definition*. In the Model Theory of Modules, they have been studied, most notably by Baur, as the *pp-definable subgroups* of M_R.

The assignment $M \mapsto \varphi(M)$ is functorial in the sense that if $f : M \to N$ is an R-morphism, then $f(\varphi(M)) \subseteq \varphi(N)$, which induces a morphism $\varphi(f) : \varphi(M) \to \varphi(N)$ of abelian groups. Restricted to mod-R, this defines a subfunctor of $(R_R, -)$, which we denote by $\varphi(-)$, or simply φ.

Proposition 1.2 (see [17, Corollary 12.4]). *A subfunctor of the forgetful functor* $(R_R, -)$ *is finitely generated iff* $F = \varphi(-)$ *for some right pp-formula* $\varphi(x)$ *over* R.

Proof. Let $\varphi(x)$ be a right pp-formula over R of the form (1). Let M_R be the finitely presented right R-module with free presentation

$$R_R^n \xrightarrow{g} R_R^{1+m} \xrightarrow{p} M_R \to 0 \tag{2}$$

where g denotes multiplication on the left by the matrix $\binom{A}{B}$. Let $e_0, e_1, \ldots, e_m$ be the canonical basis for R^{1+m}. We will use the Generator Lemma to show that $a = p(e_0) \in M$ is a generator of $\varphi(-)$: Suppose $X_R \in$ Mod-R and $b \in \varphi(X)$. Take $c_1, \ldots, c_m \in X$ such that

$$(b, c_1, \ldots, c_m)\begin{pmatrix} A \\ B \end{pmatrix} = \mathbf{0}$$

and let $f : R^{1+m} \to X$ be a morphism such that $f(e_0) = b$ and for each $i \le m$, $f(e_i) = c_i$.

$$\begin{tikzcd} R^n \arrow[r, "g"] \arrow[dr, "0"'] & R^{1+m} \arrow[r, "p"] \arrow[d, "f"] & M \arrow[dl, "h"] \\ & X & \end{tikzcd}$$

By hypothesis, the composition $fg = 0$, so that we can complete the diagram with $h : M_R \to X_R$, $f = hp$. It follows that $h(a) = hp(e_0) = f(e_0) = b$ and hence that $a \in M$ is a generator of $\varphi(-)$.

Conversely, if $F \subseteq (R_R, -)$ is finitely generated with generator $a \in M$, then there are natural numbers m and n and a free presentation (2) of M_R such that $p(e_0) = a$. Retracing the argument above, we can recover a right pp-formula $\varphi(x)$ over R whose corresponding functor $\varphi(-)$ is generated by $a \in M$. It follows that $F = \varphi$. □

If $a \in \varphi(M)$ is a generator of the functor $\varphi(-)$, then $a \in M$ is called a *free realization* [17, Proposition 8.12] of $\varphi(x)$. Proposition 1.2 may be used to verify that the finitely generated subfunctors of $(R_R, -)$ form a lattice: Let $\varphi, \psi \subseteq (R_R, -)$ be finitely generated subfunctors corresponding to the right pp-formulae

$$\varphi(x) \rightleftharpoons \exists \mathbf{y}(x, \mathbf{y})\begin{pmatrix} A \\ B \end{pmatrix} \doteq \mathbf{0} \quad \text{and} \quad \psi(x) \rightleftharpoons \exists \mathbf{z}(x, \mathbf{z})\begin{pmatrix} A' \\ B' \end{pmatrix} \doteq \mathbf{0},$$

respectively. As the bound tuples $\mathbf{y}$ and $\mathbf{z}$ consist of dummy variables, we may assume they have no entries in common. Defining $(\varphi \wedge \psi)(x)$ to be the right pp-formula

$$\exists \mathbf{y}, \mathbf{z}(x, \mathbf{y}, \mathbf{z}) \begin{pmatrix} A & A' \\ B & \mathbf{0} \\ \mathbf{0} & B' \end{pmatrix} \doteq \mathbf{0},$$

it is clear that for each $M \in$ mod-R, $(\varphi \wedge \psi)(M) = \varphi(M) \cap \psi(M)$, so the intersection $\varphi \cap \psi$ is equal to the finitely generated functor $(\varphi \wedge \psi)(-)$. We leave it to the reader to express the right pp-formula over R which defines the functor $\varphi + \psi$.

Suppose that $\varphi(x)$ and $\psi(x)$ are right pp-formulae over R such that the equation $\varphi(-) = \psi(-)$ holds in (mod-R, Ab) and let $a \in M$ be a generator for this functor. The proof of Proposition 1.2 indicates that for any right R-module $X \in$ Mod-R, we have $\varphi(X) = \psi(X)$. This, together with Gödel's Completeness Theorem yields the following:

Proposition 1.3. *Given two right pp-formulae $\varphi(x)$ and $\psi(x)$ over R, the following are equivalent:*

(1) $\varphi(-) = \psi(-)$.

(2) *For each* $X \in$ Mod-R, $\varphi(X) = \psi(X)$.

(3) $T_R \vdash (\forall x)(\varphi(x) \leftrightarrow \psi(x))$, *that is, there is a first-order proof in the language $\mathcal{L}_R$ of the sentence $(\forall x)(\varphi(x) \leftrightarrow \psi(x))$ from the axioms T_R for a right R-module.* $\square$

More generally, the arguments above show that the inclusion $\varphi(-) \subseteq \psi(-)$ in (mod-R, Ab) is equivalent to $T_R \vdash (\forall x)(\varphi(x) \rightarrow \psi(x))$.

All that we have done for right R-modules may be done on the left side as well [17, Section 8.4]. The language for left R-modules ${}_R\mathcal{L} = \langle +, -, 0, r \rangle_{r \in R}$ has the same symbols as $\mathcal{L}_R$. The unary function symbols r, however, act on the left. The left pp-formulae over R are then just existentially quantified systems of linear equations with the coefficients (from R) acting on the left. As in Proposition 1.2, the finitely generated subfunctors of $({}_RR, -) \in$ (R-mod, Ab) are of the form $\varphi(-)$ where $\varphi(x)$ is a left pp-formula over R. Prest and, independently, Zimmermann-Huisgen and Zimmermann have related the left and right pp-formulae over R by means of the following definition.

Definition. Let $\varphi(x) \rightleftharpoons (\exists \mathbf{y})(x, \mathbf{y})\binom{A}{B} \doteq \mathbf{0}$ be a right pp-formula over R; so that B is an $m \times n$ matrix and A is a row vector with n entries. Define the left pp-formula $\varphi^*(x)$ over R as

$$\exists \mathbf{z} \begin{pmatrix} 1 & A \\ \mathbf{0} & B \end{pmatrix} \begin{pmatrix} x \\ \mathbf{z} \end{pmatrix} \doteq \mathbf{0}$$

where $\mathbf{z} = (z_1, \ldots, z_n)$ is the n-tuple of existentially bound variables and the $\mathbf{0}$ below the 1 is an $m \times 1$ column vector.

This is well-defined on the level of functors: If $\varphi(-) = \psi(-)$, then $\varphi^*(-) = \psi^*(-)$ [17, Theorem 8.21]. More generally, the operation $\varphi \mapsto \varphi^*$ is inclusion reversing: If $\psi \subseteq \varphi$ are subfunctors of $(R_R, -)$, then $\varphi^* \subseteq \psi^*$ holds in $(R\text{-mod}, \mathrm{Ab})$.

Starting with a left pp-formula $\varphi(x)$ over R

$$\varphi(x) \rightleftharpoons (\exists \mathbf{y})(C, D)\begin{pmatrix} x \\ \mathbf{z} \end{pmatrix} \doteq \mathbf{0},$$

we can define the right pp-formula over R

$$\varphi^*(x) \rightleftharpoons (\exists \mathbf{z})(x, \mathbf{z})\begin{pmatrix} 1 & \mathbf{0} \\ C & D \end{pmatrix} \doteq \mathbf{0}.$$

This operation has properties similar to its homonym and it is easy to check that if $\varphi(x)$ is a left or right pp-formula over R, then $\varphi^{**}(-) = \varphi(-)$. It follows that the operations $\varphi \mapsto \varphi^*$ are mutual inverses and that the lattice of finitely generated subfunctors $(R_R, -)$ is anti-isomorphic to that of the finitely generated subfunctors of $({}_RR, -)$. This anti-isomorphism is called *pp-duality*. The next result, due to Zimmermann-Huisgen and Zimmermann, clarifies the relationship between the functor $\varphi(-) \in (\text{mod-}R, \mathrm{Ab})$ and its *pp-dual* $\varphi^*(-)$.

Proposition 1.4 (see [28, Lemma 2, Proposition 3]). *Let ${}_{R'}M_R$ be an R'–R-bimodule such that M_R is a finitely presented right R-module. If ${}_{R'}E$ is a left R'-module, then $\mathrm{Hom}_{R'}({}_{R'}M_R, {}_{R'}E)$ has the structure of a left R-module given by $(rf)(m) = f(mr)$. Given a right pp-formula $\varphi(x)$ over R, the finite matrix subgroup $\varphi(M)$ of M_R is an R'-submodule of ${}_{R'}M$ and if ${}_{R'}E$ is injective, then*

$$\begin{aligned}\varphi^*(\mathrm{Hom}_{R'}({}_{R'}M_R, {}_{R'}E)) &= \{f : {}_{R'}M \to {}_{R'}E : f(\varphi(M)) = 0\} \\ &\cong \mathrm{Hom}_{R'}({}_{R'}(M/\varphi(M)), {}_{R'}E).\end{aligned}$$

If, furthermore, ${}_{R'}E$ is a cogenerator, then the map

$$\varphi(M) \mapsto \varphi^*(\mathrm{Hom}_{R'}({}_{R'}M_R, {}_{R'}E))$$

is an anti-isomorphism of the respective lattices of finite matrix subgroups. □

2. Projective covers and Krull–Schmidt rings

In this section we treat projective covers in the category $(\text{mod-}R, \mathrm{Ab})$ and we characterize those rings R for which every finitely generated object of $(\text{mod-}R, \mathrm{Ab})$ has a projective cover. We begin by characterizing the finitely generated projective objects as the representable objects. Most of the arguments are classical (cf. [1, Section 17]).

Let $M \in \text{mod-}R$ and let $\eta : G \to (M, -)$ be an epimorphism in $(\text{mod-}R, \mathrm{Ab})$. As $\eta_M : G(M) \to (M, M)$ is onto, choose $a \in G(M)$ such that $\eta_M(a) = 1_M$.

By Yoneda's Lemma, there is a morphism $\gamma : (M,-) \to G$ such that $\gamma(1_M) = a$. Thus $(\eta\gamma)_M(1_M) = 1_M$ and hence $\eta\gamma = 1_{(M,-)}$. It follows that η is a split-epimorphism and that $(M,-)$ is a projective object in (mod-R, Ab).

Proposition 2.1. *A finitely generated functor $F \in$ (mod-R, Ab) is projective iff it is representable. If $M_R \in$ mod-R decomposes as a sum $M = M_1 \oplus M_2$, then $(M,-) \cong (M_1,-) \coprod (M_2,-)$ and every coproduct decomposition of $(M,-)$ arises in this way from some direct sum decomposition of M_R.*

Proof. The first statement follows from the above together with the second statement. That $(M_1 \oplus M_2,-) \cong (M_1,-) \coprod (M_2,-)$ follows from the fact that the contravariant functor $\mathrm{Hom}_R(-, X_R)$ preserves split-exact sequences. If, on the other hand $(M,-) = F \coprod G$, the idempotent projection of $(M,-)$ onto F is of the form $(f,-)$ where $f \in \mathrm{End}_R M$ is idempotent. Now $M_1 = \mathrm{Im} f$ is a direct summand of M_R and $F = \mathrm{Im}(f,-) \cong (M_1,-)$. Likewise, $G \cong (\mathrm{Im}(1_M - f),-)$. □

Definition. A subfunctor $F \subseteq G$ is said to be *small* in G if whenever $F + F' = G$, then $F' = G$. A *projective cover* of a finitely generated object $H \in$ (mod-R, Ab) is an epimorphism $\eta : (M,-) \to H$ with representable domain $(M,-)$ whose kernel $\mathrm{Ker}\,\eta$ is small in $(M,-)$.

Let $\eta : (M,-) \to H$ be a projective cover and $f : M \to M$ such that the diagram

$$\begin{tikzcd} (M,-) \arrow[rr, "{(f,-)}"] \arrow[dr, "\eta"'] & & (M,-) \arrow[dl, "\eta"] \\ & H & \end{tikzcd}$$

commutes. It is clear then that $\mathrm{Ker}\,\eta + \mathrm{Im}(f,-) = (M,-)$ so by the definition of a projective cover $(f,-)$ must be an epimorphism. Indeed, as $(M,-)$ is projective $(f,-)$ is a split-epimorphism. Now $\mathrm{Ker}(f,-) \subseteq \mathrm{Ker}\,\eta$ is small in $(M,-)$ and does not contain a coproduct factor of $(M,-)$. Hence f is an isomorphism.

We will call a generator $a \in H(M)$ a *projective cover* if the corresponding epimorphism $\eta : (M,-) \to H$, $\eta_M(1_M) = a$, is a projective cover of H. For $H = \varphi(-)$ a finitely generated subfunctor of the forgetful functor, the projective cover $a \in M$ of $\varphi(-)$ has been studied by Prest [17, Section 11.3] in the guise of "finite hull of $\varphi(x)$". The commutativity of the diagram above is tantamount to the following statement: If $a \in H(M)$ is a projective cover and $f \in \mathrm{End}_R M$ is such that $H(f)(a) = a$, then f is an isomorphism.

Proposition 2.2. *Let $H \in$ (mod-R, Ab) be finitely generated and let $a \in H(M)$ be a projective cover of H. If $f : M_R \to N_R$ is a morphism in* mod-R *such that $H(f)(a) \in H(N)$ is another generator of H, then f is a split-monomorphism. If $g : N_R \to M_R$ is a morphism in* mod-R *and there is a generator $b \in H(N)$ of H such that $H(g)(b) = a$, then g is a split-epimorphism.*

Proof. Let $f : M_R \to N_R$ be given as stated. By the Generator Lemma, there is a morphism $h : N_R \to M_R$ such that $H(hf)(a) = H(h)H(f)(a) = a$. Hence $hf : M_R \to M_R$ is an isomorphism and f is a split-epimorphism. The second statement is proved similarly. □

Suppose that $N \in$ mod-R is indecomposable and that every proper quotient of the indecomposable projective $(N,-)$ has a projective cover. If $F \subset (N,-)$, then the quotient map $(N,-) \to (N,-)/F$ must be a projective cover and, therefore, every proper subfunctor of $(N,-)$ is small in $(N,-)$. Consequently, the sum of two proper subfunctors of $(N,-)$ is again proper. Consider the subfunctor

$$\mathrm{Rad}(N,-) = \sum\{F \mid F \subset (N,-)\}.$$

Because $(N,-)$ is finitely generated, $\mathrm{Rad}(N,-)$ is a proper subfunctor of $(N,-)$ which contains all other proper subfunctors. We call a finitely generated functor F *local* if it contains a proper subfunctor $\mathrm{Rad}\, F$ which contains all other proper subfunctors of F.

Definition. A finitely presented R-module M is called *strongly indecomposable* if the functor $(M,-)$ is local.

If $(M,-)$ is local then, by the Generator Lemma, the subgroup $\mathrm{Rad}(M,M)$ consists of the non-units of (M,M) and so $\mathrm{End}_R M$ is a local ring. Conversely, if $\mathrm{End}_R M$ is local, then the map $X_R \mapsto \{f \in (M,X) \mid f$ is not a split-monomorphism$\}$ is a functor equal to $\mathrm{Rad}(M,-)$. Therefore, a finitely presented R-module is strongly indecomposable iff $\mathrm{End}_R M$ is a local ring.

Theorem 2.3. *The following are equivalent for a ring R:*

(1) *Every finitely generated object F of the category* (mod-R, Ab) *has a projective cover.*

(2) *The ring R is semiperfect and every $M \in$* mod-R *is a direct sum $M = \bigoplus_{i=1}^{n} M_i$ of strongly indecomposable modules M_i.*

(3) *Every finitely generated projective object of* (mod-R, Ab) *is a coproduct of local functors.*

Proof. The proof of the equivalence (1) ⇔ (3) is classical [1, Sections 17 and 24] and the implication (2) ⇒ (3) follows from the preceeding discussion and Proposition 2.1. We will show how Condition (1) implies that the ring R is semiperfect; then every finitely presented right R-module is a direct sum of indecomposables each of which must be strongly indecomposable.

So assume Condition (1) and let $_RX$ be a finitely generated left R-module. To show that R is semiperfect, it suffices to produce a projective cover of $_RX$ in the category R-Mod. Let $p : {_RR^n} \to {_RX}$ be an epimorphism of left R-modules. As the functor $M \otimes_R -$ is right exact for each $M \in$ mod-R, this induces an epimorphism $- \otimes p : - \otimes_R R^n \to - \otimes_R X$ in the category (mod-R, Ab). Now $- \otimes_R R^n \cong (R^n_R, -)$ is representable so that $- \otimes_R X$ is finitely generated. By hypothesis, $- \otimes_R X$ has a projective cover $F \to - \otimes_R X$, whose domain is a coproduct factor of $- \otimes_R R^n \cong F \coprod G$. Then $_RR^n = {_RF(R)} \oplus {_RG(R)}$ implies that $_RF(R)$ is a projective left R-module and $F \cong - \otimes_R F(R)$. Now it is straightforward to verify that the R-epimorphism $p|_{F(R)} : {_RF(R)} \to {_RX}$ is a projective cover of $_RX$. □

As the proof shows, insisting that R be semiperfect is unnecessary in Condition (2) of Theorem 2.3. Nevertheless, we emphasize that R is semiperfect because Warfield [23, Section 2] showed that for such a ring R, the Auslander–Bridger Transpose [5] may be defined in such a way so that it becomes a bijective correspondence between the finitely presented right R-modules and the finitely presented left R-modules (up to isomorphism). Furthermore, this Transpose commutes with direct sums and respects strong indecomposibility. Condition (2) of Theorem 2.3 thus holds for R iff the analogous statement for left R-modules holds. Consequently, the equivalent conditions of Theorem 2.3 are left-right symmetric, that is, they are true of the ring R iff they are true of the opposite ring R^{op}.

Definition. A ring R is called *Krull–Schmidt* if it satisfies any of the equivalent conditions of Theorem 2.3 or, equivalently, any one of their left counterparts.

The following important correspondence, which holds in the context of Krull–Schmidt rings, pervades the sequel. We will state it for right R-modules only. The proof is immediate from the existence and uniqueness of projective covers and Yoneda's Lemma.

Theorem 2.4. *Let R be a Krull–Schmidt ring. The assignment*

$$M_R \mapsto S_M = (M, -)/\operatorname{Rad}(M, -)$$

is a bijective correspondence between the finitely presented indecomposable right R-modules M_R and the simple functors $S \in$ (mod-R, Ab). Given such a functor S, then $S = S_M$ where M_R is the unique indecomposable in mod-R *such that $S(M) \neq 0$.* □

Theorem 2.4 may be used to distinguish a special class of finitely presented indecomposable modules over a Krull–Schmidt ring: Call an indecomposable $M \in \text{mod-}R$ *isolated* if $\mathrm{Rad}(M,-)$ is a finitely generated functor. Let M_R be isolated and let $a \in M$, $a \neq 0$ generate the subfunctor φ of the forgetful functor. The epimorphism $\eta : (M,-) \to \varphi(-)$ determined by $\eta_M(1_M) = a$ is then a projective cover of $\varphi(-)$. By Proposition 1.2, the functor $\eta(\mathrm{Rad}(M,-))$ is of the form $\psi(-)$ for some right pp-formula over R and so the simple functor $S_M = (M,-)/\mathrm{Rad}(M,-) \cong \varphi/\psi$. The pair of functors $\psi \subset \varphi$ has the property that if σ is any (finitely generated) functor such that $\psi \subseteq \sigma \subseteq \varphi$, then $\sigma = \psi$ or $\sigma = \varphi$; such a pair of finitely generated subfunctors of the forgetful functor is called a *minimal pair.*

Proposition 2.5. *A finitely presented indecomposable right R-module M_R is isolated iff there is a minimal pair $\psi \subset \varphi$ of finitely generated subfunctors of the forgetful functor $(R_R,-)$ such that $S_M \cong \varphi/\psi$.*

Proof. One direction has already been proved. Assume that $M \in \text{mod-}R$ is indecomposable and that a minimal pair $\psi \subset \varphi$ exists such that $S_M \cong \varphi/\psi$. Let $a \in \varphi(M) \setminus \psi(M)$ generate the functor φ'. Then $S_M \cong \varphi'/(\varphi' \wedge \psi)$ where $(\varphi' \wedge \psi) \subset \varphi'$ is another minimal pair. The epimorphism $\eta : (M,-) \to \varphi'$ defined by $\eta_M(1_M) = a$ is part of the short exact sequence

$$0 \to \mathrm{Ker}\,\eta \hookrightarrow \mathrm{Rad}(M,-) \xrightarrow{\eta} (\varphi' \wedge \psi) \to 0.$$

As $(\varphi' \wedge \psi)$ is finitely generated, and our goal is to prove $\mathrm{Rad}(M,-)$ also to be such, it suffices to verify that $\mathrm{Ker}\,\eta$ is finitely generated. Think of η as a morphism from $(M,-)$ to $(R,-)$ so that $\eta = (f,-)$ for some $f : R_R \to M_R$ (in fact $f(1) = a$). If $C = \mathrm{Coker}\,f$, then by the right exactness of the functor $\mathrm{Hom}_R(-,X_R)$ the exact sequence

$$R \xrightarrow{f} M \to C \to 0$$

induces an exact sequence in $(\text{mod-}R, \mathrm{Ab})$

$$0 \to (C,-) \to (M,-) \xrightarrow{\eta} (R,-)$$

showing that $\mathrm{Ker}\,\eta$ is finitely generated. □

Let N_R be a finitely presented indecomposable right module over a Krull–Schmidt ring R; then $R' = \mathrm{End}_R\, N_R$ is a local ring. If ${}_{R'}E$ is the injective hull of the unique simple left R'-module, then ${}_{R'}E$ is an injective cogenerator and the full force of Proposition 1.4 may be applied to the left R-module

$${}_R(N^\sharp) = \mathrm{Hom}_{R'}({}_{R'}N_R, {}_{R'}E). \tag{3}$$

If N_R is isolated with $S_N \cong \varphi/\psi$, then by Proposition 1.4, $(\psi^*/\varphi^*)({}_RN^\sharp) \neq 0$. Now ${}_RN^\sharp$ has a local endomorphism ring [27, p. 312] and $\psi^* \subset \varphi^*$ is a minimal pair of finitely generated subfunctors of $({}_RR,-)$, so if ${}_RN^\sharp$ is finitely presented, then it must be isolated.

Corollary 2.6. *Let R be a Krull–Schmidt ring and N_R a finitely presented isolated right R-module with $S_N \cong \varphi/\psi$. If the left R-module ${}_RN^\sharp$ defined by* (3) *is finitely presented, then ${}_RN^\sharp$ is isolated and $S_{N^\sharp} \cong (\psi^*/\varphi^*)$.* □

3. Left pure-semisimple rings

A ring R is called *left pure-semisimple* if the subfunctors of $(R_R,-) \in (\text{mod-}R,\text{Ab})$ satisfy the ascending chain condition. Since every functor $F \in (\text{mod-}R,\text{Ab})$ is the sum of its finitely generated subfunctors (by Yoneda's Lemma), this is equivalent to the condition that every subfunctor of $(R_R,-) \in (\text{mod-}R,\text{Ab})$ is finitely generated. By Proposition 1.2, the ring R is left pure-semisimple iff every subfunctor of $(R_R,-)$ is of the form $\varphi(-)$ where φ is a right pp-formula.

Let us show that if R is left pure-semisimple, then every finitely generated functor $F \in (\text{mod-}R,\text{Ab})$ satisfies the ascending chain condition on subfunctors. By Proposition 1.1, it is enough to verify this for every representable functor $(M_R,-)$ with $M \in \text{mod-}R$. There is an epimorphism $f : R_R^n \to M_R$ from some finitely generated free module to M_R. This induces a monomorphism of functors $(f,-) : (M,-) \to (R^n,-) \cong (R,-)^n$. Since the ascending chain condition clearly holds for subfunctors of $(R,-)^n$, it must hold for $(M,-)$.

The characterizations of left pure-semisimple rings are legion. Those which will be useful to us are included in the next theorem.

Theorem 3.1 (see [3,4,8,14,26]). *Every left pure-semisimple ring is left artinian. A ring R is left pure-semisimple iff one of the following equivalent conditions holds*:

(1) *Every left R-module is a direct sum of indecomposable modules.*
(2) *Every left R-module is a direct sum of finitely generated modules.*
(3) *The functor $({}_RR,-) \in (R\text{-mod},\text{Ab})$ satisfies the descending chain condition on finitely generated subfunctors.* □

This last characterization of left pure-semisimple rings is an immediate consequence of pp-duality. As every left pure-semisimple ring R is left artinian, we see that the finitely presented left R-modules are precisely the finitely generated left R-modules. It also follows that every left pure-semisimple ring is Krull–Schmidt and hence that every finitely generated functor $F \in (R\text{-mod},\text{Ab})$ is a finite sum of local functors; we will need this in the next proof.

Definition (see [3, p. 290]). A sequence of morphisms $\mathcal{F} = \{f_i : M_i \to M_{i+1}\}_{i<\omega}$ of left R-modules is *noetherian* if there is a natural number $n < \omega$ such that $f_n \cdots f_1 f_0 = 0$ or for every $k \geq n$, the morphism f_k is an isomorphism.

The following characterization of left pure-semisimple rings is well-known. We prove only one direction as an illustration of the techniques used in the sequel.

Lemma 3.2 (see [3,4,11,15]). *Let R be a Krull–Schmidt ring. Then R is left pure-semisimple iff every sequence of morphisms $\mathcal{F} = \{f_i : M_i \to M_{i+1}\}_{i<\omega}$ of (finitely presented) indecomposable left R-modules is noetherian.*

Proof. We will just prove that under this noetherian condition R is left pure-semisimple. Suppose that the descending chain condition fails for the finitely generated subfunctors of $({}_RR, -)$. Call a finitely generated subfunctor φ of $({}_RR, -)$ *unfounded* if it contains a strictly descending chain of finitely generated subfunctors; so our assumption is that $({}_RR, -)$ is unfounded. Clearly, every unfounded φ strictly contains another unfounded ψ and if we decompose an unfounded φ as a finite sum $\varphi = \sum_{i \in I} \varphi_i$ of local subfunctors φ_i, then one of the φ_i must be unfounded as well. Thus arises a strictly descending sequence of local subfunctors of $({}_RR, -)$:

$$\varphi_0 \supset \varphi_1 \supset \cdots \supset \varphi_n \supset \cdots.$$

For each $i < \omega$, let $a_i \in \varphi_i(M_i)$ be a projective cover of $\varphi_i(-)$. Every M_i is therefore indecomposable. By the Generator Lemma there is a morphism $f_i : M_i \to M_{i+1}$ such that $f(a_i) = a_{i+1}$. It is clear from the definition of the f_i that for each $n < \omega$, the composition $f_n \cdots f_1 f_0 \neq 0$. And because the descending chain is strict, no f_i is an isomorphism. $\square$

Let R and R' be rings. An equivalence between the category of finitely generated right R-modules and the category opposite of the finitely generated left R'-modules is called a *Morita duality.* A necessary condition for such a duality to exist is that R be right artinian and that R' be left artinian. Thus the categories in question are mod-R and R'-mod and Morita duality is a contravariant functor $\mathcal{D}$ between the two.

A ring R (respectively R') is called *right (respectively left) Morita* if there exists a ring R' (respectively R) and a Morita duality $\mathcal{D}$: mod-$R \to R'$-mod. A necessary and sufficient condition for R to be right Morita is that R be right artinian and that there be a finitely generated injective cogenerator E_R in the category Mod-R of right R-modules. If $R' = \operatorname{End}_R E_R$, then the contravariant functor $\mathcal{D} = \operatorname{Hom}_R(-, {}_{R'}E_R)$: mod-$R \to R'$-mod is a Morita duality. There is, of course, an analogous characterization of left Morita rings from which it is easy to spot that every left pure-semisimple ring is left Morita. The following theorem is due to Simson.

Theorem 3.3 (see [20]). *Let $\mathcal{D}$: mod-$R \to R'$-mod be a Morita duality. Then R is left pure-semisimple iff R' is left pure-semisimple.*

This will follow from a result of Zimmermann-Huisgen and Zimmermann just as soon as we introduce another definition.

Definition (see [3, p. 290]). A sequence of morphisms $\mathcal{G} = \{g_i : N_{i+1} \to N_i\}_{i<\omega}$ of left R-modules is *artinian* if there is a natural number $n < \omega$ such that $g_0 g_1 \cdots g_n = 0$ or for every $k \geq n$, the morphism g_k is an isomorphism.

Proposition 3.4 (see [28, Lemma Tr] and [11, Proposition]). *Let R be a semiprimary ring, that is, the Jacobson radical $J(R)$ of R is nilpotent and $R/J(R)$ is semisimple artinian. Then the following are equivalent:*
(1) *Every sequence of morphisms $\mathcal{F} = \{f_i : M_i \to M_{i+1}\}_{i<\omega}$ of finitely presented indecomposable left R-modules is noetherian.*
(2) *Every sequence of morphisms $\mathcal{G} = \{g_i : N_{i+1} \to N_i\}_{i<\omega}$ of finitely presented indecomposable right R-modules is artinian.* □

Proof of Theorem 3.3. Let $\mathcal{D}$: mod-$R \to R'$-mod be a Morita duality. Then both of the rings R and R' are Krull–Schmidt and semiprimary. By Lemma 3.2, the ring R is left pure-semisimple iff every chain of morphisms between finitely presented indecomposable left R-modules is noetherian. By Proposition 3.4, this is equivalent to every chain of morphisms between finitely presented indecomposable right R-modules being artinian which, by the Morita duality, is equivalent to every chain of morphisms between finitely presented indecomposable left R'-modules being noetherian. Finally, this last clause is equivalent, by Lemma 3.2, to R' being left pure-semisimple. □

Suppose that R is left pure-semisimple and that N_R is a finitely presented indecomposable right R-module. Because every subfunctor of $(N,-) \in$ (mod-R, Ab) is finitely generated, N_R is isolated. The indecomposable left R-module ${}_RN^\sharp$ defined by (3) is finitely generated and hence, by Corollary 2.6, it is isolated.

Proposition 3.5. *Let R be left pure-semisimple. The rule $N_R \mapsto {}_RN^\sharp$ is a bijective correspondence between the finitely presented indecomposable right R-modules and the isolated left R-modules.*

Proof. The only point that needs to be clarified is that this assignment is surjective. If $X \in R$-mod is isolated, then the corresponding simple functor $S_X \cong \varphi/\psi$ where $\psi \subset \varphi$ is a minimal pair of finitely generated subfunctors of $({}_RR, -)$. If $N_R \in$ mod-R is such that $S_N \cong \psi^*/\varphi^*$, then by Corollary 2.6, $X \cong N^\sharp$. $\square$

If R is a Krull–Schmidt ring and $N_R \in$ mod-R is indecomposable, then there is a ring isomorphism between the local ring $R_N = \mathrm{End}_R\, N_R$ and the endomorphism ring of ${}_RN^\sharp$ [27, p. 312]. It is shown in [16, Corollary 2.5] that for each indecomposable $N \in$ mod-R, the rule $N \mapsto N^\sharp$ is part of a Morita duality $\mathcal{D}_N : R_N$-mod $\to$ mod-R_N. It is for this reason that the correspondence $N \to N^\sharp$ is called *endoduality* and that the left R-module ${}_RN^\sharp$ is called the *endodual* of N_R. The existence of this endoduality essentially depends on the following result.

Proposition 3.6 (see [16, Theorem 2.3]). *Let R be a left pure-semisimple ring. Every finitely presented right R-module M_R is endofinite, that is, M has finite length as an* $\mathrm{End}_R\, M_R$*-module. Every isolated left R-module is endofinite.* $\square$

The second statement follows from the first together with Proposition 1.4 and the following easy consequence of the Generator Lemma.

Proposition 3.7. *Let R be any ring and M_R a finitely presented right R-module. Then M_R is endofinite iff there is a composition series in the lattice of finite matrix subgroups of M_R. When this is the case, an abelian subgroup of M is a finite matrix subgroup of M_R iff it is an* $\mathrm{End}_R\, M_R$*-module.* $\square$

4. Preinjective modules

In this section, we will consider the modules called *preinjective*. Particular attention will be paid to their behavior under Morita duality and endoduality. The study of such modules was initiated by Auslander and Smalø [6] in the context of artin algebras. Over a left pure-semisimple ring they were studied by Zimmermann-Huisgen [27].

A subcategory of mod-R is called *additive* if it is closed under isomorphism, direct sums and direct summands. If $\mathcal{C}$ is a subcategory of mod-R, then $\mathrm{add}\,\mathcal{C}$ denotes the smallest additive subcategory of mod-R containing $\mathcal{C}$. That a category is additive is expressed by the equation $\mathcal{C} = \mathrm{add}\,\mathcal{C}$. If $\mathcal{C} = \{M\}$ consists of a single object, then $\mathrm{add}\,\mathcal{C}$ will be denoted by $\mathrm{add}(M)$. If $\mathcal{C} = \mathrm{add}\,\mathcal{C} \subseteq$ mod-R, define mod-$R \mid \mathcal{C}$ to be the subcategory of mod-R consisting of those objects without a summand in $\mathcal{C}$.

Let R be a Krull–Schmidt ring and $M_R \in$ mod-R. Then $\mathrm{add}(M)$ is the category of finitely presented right R-modules all of whose indecomposable summands are also summands of M_R. The category mod-$R \mid \mathrm{add}(M)$ is then additive. If $F \in$ (mod-R, Ab) has finite length, consider a composition series

$$F = F_0 \supset F_1 \supset \cdots \supset F_m = 0.$$

Every composition factor F_i/F_{i+1} is a simple functor corresponding to some indecomposable $N_i \in \text{mod-}R$, so if $K \in \text{mod-}R$ is an indecomposable such that $F(K) \neq 0$, then for some i, $K \cong N_i$. Letting $N = \bigoplus_i N_i$, we have that $F = 0$ when restricted to the subcategory $\text{mod-}R \mid \text{add}(N)$.

Assumption. *For the remainder of this section R will denote a left pure-semisimple ring.*

Proposition 4.1. *Let R be left pure-semisimple. A finitely generated functor $\varphi \subseteq (R_R, -)$ has finite length iff there is a finitely presented module $M_R \in \text{mod-}R$ such that $\varphi(K) = 0$ for all $K_R \in \text{mod-}R \mid \text{add}(M)$.*

Proof. Let $M \in \text{mod-}R$ be such that $\varphi(K) = 0$ for all $K \in \text{mod-}R \mid \text{add}(M)$. We will show that the lattice of (finitely generated) subfunctors of φ is isomorphic to the lattice of finite matrix subgroups of M_R contained in $\varphi(M_R)$, which has a composition series by Proposition 3.7. The map $\psi \mapsto \psi(M)$ is clearly onto and if $\psi_1 \subset \psi_2 \subseteq \varphi$, then there is a finitely presented indecomposable $N_R \in \text{mod-}R$ such that $(\psi_2/\psi_1)(N_R) \neq 0$. By hypothesis N_R is isomorphic to a summand of M_R and hence $\psi_1(M) \subset \psi_2(M)$. $\square$

By the ascending chain condition on the subfunctors of $(R_R, -)$, there is a maximal subfunctor $(R_R, -)_0$ of finite length; it contains all other finite length subfunctors. We will study the support of this functor.

Definition. An indecomposable module $N_R \in \text{mod-}R$ is called *preinjective* if there is a finitely presented right R-module M_R such that $N \in \text{mod-}R \mid \text{add}(M)$ and whenever the morphism $f : N_R \to K_R$ is a monomorphism with $K \in \text{mod-}R \mid \text{add}(M)$, then f is a split-monomorphism. We will say that M_R *witnesses* the preinjectivity of N_R.

Theorem 4.2. *Let R be left pure-semisimple and right artinian. A finitely presented indecomposable $N \in \text{mod-}R$ is preinjective iff $(R_R, N)_0 \neq 0$.*

Proof. Let $N_R \in \text{mod-}R$ be preinjective and let $M_R \in \text{mod-}R$ be a witness to this. Consider the reject of N_R in $\text{mod-}R \mid \text{add}(M \oplus N)$:

$$\text{Rej}(N_R) = \bigcap\{\text{Ker} f \mid f : N \to Y \text{ and } Y \in \text{mod-}R \mid \text{add}(M \oplus N)\}.$$

As R is right artinian, N_R has finite length and this intersection may be taken over a finite subset. By hypothesis, $\text{Rej}(N_R) \neq 0$. Pick $a \in \text{Rej}(N_R)$ which is nonzero and let φ be the finitely generated subfunctor of $(R_R, -)$ generated by $a \in N_R$. Since the $\text{Rej}(N_R)$ is an $\text{End}_R N$-module, $\varphi(N) \subseteq \text{Rej}(N)$. By the

Generator Lemma, $\varphi(K) = 0$ for each $K_R \in \text{mod-}R \mid \text{add}(M \oplus N)$ and so by Proposition 4.1, φ has finite length.

For the converse, let φ be a minimal functor of finite length with the property that $\varphi(N) \neq 0$. By Yoneda's Lemma, there is a nonzero morphism $\eta : (N, -) \to \varphi$. As $\text{Im}\,\eta \subseteq \varphi$ and $(\text{Im}\,\eta)(N) \neq 0$ it must be that η is an epimorphism and, therefore, a projective cover. There are only finitely many indecomposables $M_i \in \text{mod-}R$ such that $(\text{Rad}\,\varphi)(M_i) \neq 0$. Let $M = \bigoplus_i M_i$. We claim that M_R witnesses the preinjectivity of N_R. First, $N \in \text{mod-}R \mid \text{add}(M)$ since, by the choice of φ, $(\text{Rad}\,\varphi)(N) = 0$. Second, let $K \in \text{mod-}R \mid \text{add}(M)$ and $f : N \to K$ a monomorphism. As $a = \eta_N(1_N) \in \varphi(N)$ is a projective cover and $f(a) \in \varphi(K) \setminus (\text{Rad}\,\varphi)(K)$ is a generator of φ, the monomorphism f splits, by Proposition 2.2. □

Thus there are only finitely many preinjective right R-modules. Indeed, a finitely presented indecomposable right R-module is preinjective iff it corresponds, via Theorem 2.4 to a composition factor of $(R_R, -)_0$.

Definition. An indecomposable module ${}_RY \in R\text{-mod}$ is called *preprojective* if there is a finitely presented left R-module ${}_RX$ such that $Y \in R\text{-mod} \mid \text{add}(X)$ and if the morphism $f : {}_RZ \to {}_RY$ is an epimorphism with $Z \in R\text{-mod} \mid \text{add}(X)$, then f is a split-epimorphism.

The notion of a preprojective module is the categorical dual of the notion of preinjective. More precisely, if $\mathcal{D} : \text{mod-}R \to R'\text{-mod}$ is a Morita duality, then a module $N_R \in \text{mod-}R$ is preinjective iff the module $\mathcal{D}(N) \in R'\text{-mod}$ is preprojective. Since every left pure-semisimple ring is left Morita, the following is a consequence of Theorems 3.3 and 4.2.

Corollary 4.3. *If R is left pure-semisimple, then there are only finitely many preprojective left R-modules.*

The relationship between the preinjective right R-modules and the preprojective left R-modules is, as we shall see, more interesting with regard to endoduality. Call a subfunctor $F \subseteq ({}_RR, -)$ *cofinite in* $({}_RR, -)$ if the quotient functor $({}_RR, -)/F$ has finite length.

Theorem 4.4. *Let R be left pure-semisimple and $F \in (R\text{-mod}, \text{Ab})$ cofinite in $({}_RR, -)$. If ${}_RY$ is a finitely presented indecomposable left R-module with the property that $Y \supset F(Y)$, then ${}_RY$ is preprojective.*

Proof. Let $F \subseteq ({}_RR, -)$ be a maximal functor cofinite in $({}_RR, -)$ such that $Y \supset F(Y)$. By the definition, F cannot be written as a nontrivial intersection $F = F_1 \cap F_2$ of properly larger subfunctors of $({}_RR, -)$. Thus the finite length functor $G = ({}_RR, -)/F$ has a unique simple subfunctor necessarily isomorphic to S_Y. There are only finitely many indecomposables $X_i \in R$-mod such that $(G/S_Y)({}_RX_i) \neq 0$. Let ${}_RX = \bigoplus_i X_i$. We claim that ${}_RX$ witnesses the preprojectivity of ${}_RY$. By the choice of F, it is clear that $(G/S_Y)(Y) = 0$, so that ${}_RY \notin \text{add}({}_RX)$. Now suppose that $g : {}_RZ \to {}_RY$ is an epimorphism with $Z \in R\text{-mod} \mid \text{add}(X)$. Then because G is a quotient of the forgetful functor it preserves epimorphisms and the morphism of abelian groups $G(g) : G(Z) \to G(Y)$ must be an epimorphism. But $G(Z) = S_Y(Z)$ and $G(Y) = S_Y(Y)$ so that $G(g) = S_Y(g)$. Any nonzero $c \in S_Y(Y)$ is a projective cover and any nonzero $a \in S_Y(Z)$ is a generator of S_Y, so by Proposition 2.2, g is a split-epimorphism. □

By the descending chain condition on finitely generated subfunctors of $({}_RR, -)$, there is a minimal such functor $({}_RR, -)^0$ cofinite in $({}_RR, -)$. By pp-duality, $({}_RR, -)^0 = (R_R, -)^*_0$ which yields the following corollary.

Corollary 4.5. *Let R be left pure-semisimple. The endodual of a preinjective right R-module is an isolated preprojective left R-module. So if $\iota_r(R)$ is the number of preinjective right R-modules and $\wp_l(R)$ is the number of preprojective left R-modules, then the following inequality holds:*

$$\iota_r(R) \leq \wp_l(R). \qquad \square$$

5. Rings of finite representation type

A ring R is said to be of *finite representation type* if the functor $(R_R, -) \in (\text{mod-}R, \text{Ab})$ has finite length. Every ring of finite representation type is therefore left pure-semisimple. The rings of finite representation type have been characterized in many ways, some of which are included in the following.

Theorem 5.1 (see [3,4]). *The following are equivalent for a ring R:*
(1) *The ring R is of finite representation type.*
(2) *The ring R is left pure-semisimple and there is a finitely presented right R-module $M_R \in$ mod-R such that* mod-R = add(M_R).
(3) *The functor $({}_RR, -) \in (R\text{-mod}, \text{Ab})$ has finite length.*
(4) *The ring R is left pure-semisimple and there are only finitely many indecomposable left R-modules.* □

The equivalence of Conditions (1) and (2) follows from Proposition 4.1. The equivalence of Conditions (1) and (3), which is immediate from pp-duality, implies that finite representation type is a left–right symmetric condition. It is also clear from pp-duality that a left pure-semisimple ring is of finite representation type iff it is right pure-semisimple. But there is no known example of a left pure-semisimple ring which is not of finite representation type, that is, it is not known whether pure-semisimplicity is a left–right symmetric notion. For artin algebras, Auslander [4] has settled the following conjecture in the positive.

The Pure-semisimple Conjecture. Every left pure-semisimple ring is of finite representation type.

If R is of finite representation type, then Condition (4) of Theorem 5.1 follows from Condition (3) and left-right symmetry. The Condition (2) follows from Condition (4) and Proposition 3.5. When R is of finite representation type, then we have the following equalities of functors:

$$(R_R,-)_0 = (R_R,-), \qquad ({}_RR,-)^0 = 0.$$

It follows from Theorem 4.2 that every indecomposable right R-module is preinjective, while Theorem 4.4 implies that every indecomposable left R-module is preprojective.

Theorem 5.2. *A right artinian and left pure-semisimple ring R is of finite representation type iff the number of preinjective right R-modules is the same as the number of preprojective left R-modules. In the terminology of Corollary* 4.5, *this is equivalent to the equality*

$$\imath_r(R) = \wp_l(R).$$

Proof. If R is of finite representation type, then it is left and right pure-semisimple. Every indecomposable left R-module is therefore isolated. By Proposition 3.5, endoduality is a bijection between the indecomposable right R-modules and the indecomposable left R-modules. The number of left preprojectives is therefore equal to that of the right preinjectives.

For the converse, suppose that R is not of finite representation type. We will exhibit a non-isolated preprojective left R-module. The result then follows from Proposition 3.5 and Corollary 4.5. By hypothesis, we have that $({}_RR,-)^0$ is a nonzero functor in the category $(R\text{-mod}, \mathrm{Ab})$. Let $\eta : (M,-) \to ({}_RR,-)^0$ be a projective cover and let ${}_RN$ be an indecomposable summand of $M = N \oplus K$. If ${}_RN$ were isolated, then $F = \mathrm{Rad}(N,-) \coprod (K,-)$ would be a finitely generated subfunctor of $(M,-)$ and so $\eta(F)$ would be a finitely generated functor cofinite in $({}_RR,-)$, but properly contained in $({}_RR,-)^0$. That is a contradiction. That ${}_RN$ is preprojective follows from Theorem 4.4 and the fact that $N \supset \eta(F)(N)$. □

A ring R is said to have *self-duality* if there is a Morita duality $\mathcal{D}$: mod-$R \to R$-mod. Every artin algebra has self-duality. It is clear that a left pure-semisimple ring with self-duality has as many left preprojective modules as it does right preinjective modules.

Corollary 5.3. *A left pure-semisimple ring with self-duality is of finite representation type.*

Definition. A *Morita sequence* is a finite sequence of rings $(R_1, R_2, \ldots, R_n)$ such that for each $k < n$ there exists a Morita duality $\mathcal{D}_k$: mod-$R_k \to R_{k+1}$-mod.

Given a Morita sequence $(R_1, R_2, \ldots, R_n)$ with R_1 a left pure-semisimple ring, we may infer from Theorem 3.3 that for each $i \leq n$, the ring R_i is also left pure-semisimple. By Theorems 3.3 and 5.1, we have the following.

Proposition 5.4. *If $\mathcal{D}$: mod-$R \to R'$-mod is a Morita duality, then R is of finite representation type iff R' is.* □

So if the initial entry R_1 of a Morita sequence $(R_1, R_2, \ldots, R_n)$ is a counterexample to the Pure-semisimple Conjecture, then so is every subsequent ring R_i for $i \leq n$.

Proposition 5.5. *Let R be a counterexample to the Pure-semisimple Conjecture. The length n of any Morita sequence $(R = R_1, R_2, \ldots, R_n)$ is bounded by the number of preprojective left R-modules, that is, we have the inequality*

$$n \leq \wp_l(R).$$

Proof. By the Morita duality $\mathcal{D}_i$: mod-$R_i \to R_{i+1}$-mod, we have that for each $i < n$, $\iota_r(R_i) = \wp_l(R_{i+1})$. As R_1 is a counterexample to the Pure-semisimple Conjecture, so is every ensuing R_i. If $i < n$, then R_i is, in addition, right artinian so Theorem 4.5 applies to yield the inequality $\iota_r(R_i) < \wp_l(R_i)$. Combining this with the above gives that for each $i < n$,

$$\wp_l(R_{i+1}) < \wp_l(R_i).$$

Now every projective indecomposable is preprojective so $\wp_l(R_n) > 0$ and hence for every $j < n$, $\wp_l(R_{n-j}) > j$. Letting $j = n - 1$ yields the proposition. □

If $(R_1, R_2, \ldots, R_n)$ is a Morita sequence, then so is any initial segment $(R_1, R_2, \ldots, R_m)$ with $m \leq n$. The former sequence is then called an *extension* of the latter. It is clear that the Morita sequence $(R_1, R_2, \ldots, R_n)$ has a proper extension iff the final entry R_n is right Morita.

Corollary 5.6. *If there exists a counterexample to the Pure-semisimple Conjecture, then there exists a counterexample which is not right Morita.*

Proof. Let R be a counterexample and consider a Morita sequence $(R = R_1, R_2, \ldots, R_n)$ which has no proper extensions; such a sequence exists by the previous proposition. The ring R_n is then a counterexample which is not right Morita. □

A counterexample to the Pure-semisimple Conjecture is called *minimal* if for every nontrivial ideal I of R, the quotient ring R/I is of finite representation type. It is clear that if a counterexample is given, then modding out by a large enough ideal, one obtains a minimal counterexample.

Proposition 5.7. *If R is a minimal counterexample to the Pure-semisimple Conjecture and $(R = R_1, \ldots, R_n)$ is a Morita sequence, then for each $i \leq n$ the ring R_i is also a minimal counterexample.* □

This follows from the fact [1, Section 24] that if $\mathcal{D}$: mod-$R \to R'$-mod is a Morita duality, then there is an isomorphism $I \mapsto I'$ from the lattice of ideals of R to the lattice of ideals of R' and the restriction $\mathcal{D}_I = \mathcal{D}|_{\text{mod-}R/I}$ is a Morita duality $\mathcal{D}_I$: mod-$R/I \to (R'/I')$-mod. The following was inspired by Simson's work [21].

Proposition 5.8. *If R is a minimal counterexample to the Pure-semisimple Conjecture, then R contains a unique minimal two-sided ideal.*

Proof. Let $(R = R_1, \ldots, R_n)$ be a Morita sequence without a proper extension. It is enough to prove the theorem for the final entry R_n. So we may assume without loss of generality that R is not right Morita.

Suppose, to the contrary, that there are nontrivial ideals I_1 and I_2 of R such that $I_1 \cap I_2 = 0$. As R is a minimal counterexample, both of the quotient rings R/I_1 and R/I_2 are of finite representation type and hence right Morita. The monomorphism $f : R \to R/I_1 \oplus R/I_2$ of right R-modules, defined by $f(r) = (r + I_1) \oplus (r + I_2)$, implies that R is right artinian. If S_R is a simple right R-module and $E_R = E(S)$ is the injective hull of S_R, then applying the exact functor $\mathrm{Hom}_R(-, E_R)$ gives an epimorphism of right R-modules

$$\mathrm{Hom}_R(R/I_1, E_R) \oplus \mathrm{Hom}_R(R/I_2, E_R) \to \mathrm{Hom}_R(R, E_R) \cong E_R.$$

The uniform module E_R is then a sum of the image of a uniform R/I_1-module and the image of a uniform R/I_2-module. Each of these is finitely generated and therefore so is E_R. This contradicts the hypothesis that R is not right Morita. □

6. Formal triangular matrix rings

In this final section, we prove results about left pure-semisimple rings that were attained by Simson [21] in the hereditary case.

If there exists a counterexample to the Pure-semisimple Conjecture, it was shown in the previous section that then there exists a minimal counterexample R which is not right Morita. This counterexample is minimal in the sense that it contains a unique minimal ideal I and the quotient ring R/I is of finite representation type. We will demonstrate how the formal lower triangular matrix ring $(R/J(R))_I$ corresponding to the $R/J(R)$–$R/J(R)$-bimodule I is another counterexample to the Pure-semisimple Conjecture. This counterexample $(R/J(R))_I$ is not right Morita, it is (left and right) hereditary and its Jacobson radical is a minimal ideal. A general treatment of Formal triangular matrix rings is offered in [13, Section 4A].

Let J denote the Jacobson radical of R. The ring $(R/J)_I$, sometimes denoted as

$$(R/J)_I = \begin{pmatrix} R/J & 0 \\ I & R/J \end{pmatrix},$$

is the formal lower triangular matrix ring consisting of matrices

$$\begin{pmatrix} r_1 + J & 0 \\ i & r_2 + J \end{pmatrix} \quad \text{where } r_1, r_2 \in R \text{ and } i \in I.$$

Addition is defined entrywise while multiplication is given by

$$\begin{pmatrix} r_1 + J & 0 \\ i & r_2 + J \end{pmatrix} \begin{pmatrix} r_1' + J & 0 \\ i' & r_2' + J \end{pmatrix} = \begin{pmatrix} r_1 r_1' + J & 0 \\ i r_1' + r_2 i' & r_2 r_2' + J \end{pmatrix}.$$

The Jacobson radical $J((R/J)_I)$ consists of the strictly lower triangular matrices. The (left, right) ideals of $(R/J)_I$ contained in $J((R/J)_I)$ coincide with the (left, right) ideals of R contained in I. Thus $J((R/J)_I)$ is a minimal ideal. As the quotient ring $(R/J)_I/J((R/J)_I) \cong R/J \oplus R/J$ is semisimple artinian, the ring $(R/J)_I$ is left artinian.

The left $(R/J)_I$-modules are in bijective correspondence with the class of triples $(X', X''; \lambda)$ where X' and X'' are left R/J-modules and $\lambda : {}_{R/J}I \otimes_{R/J} X' \to {}_{R/J}X''$ is an R/J-morphism. The associated module may be represented as the set of column vectors $\binom{x'}{x''}$ with $x' \in X'$ and $x'' \in X''$, the action of $(R/J)_I$ given by

$$\begin{pmatrix} r_1 + J & 0 \\ i & r_2 + J \end{pmatrix} \begin{pmatrix} x' \\ x'' \end{pmatrix} = \begin{pmatrix} r_1 x' \\ \lambda(i \otimes x') + r_2 x'' \end{pmatrix}.$$

This formula leaves no choice as to how a morphism of triples $f : (X', X''; \lambda) \to (Y', Y''; \gamma)$ ought to be defined: It is a pair $f = (f', f'')$ of R/J-morphisms such that the diagram

$$\begin{array}{ccc} I \otimes X' & \xrightarrow{\lambda} & X'' \\ {\scriptstyle 1_I \otimes f'} \downarrow & & \downarrow {\scriptstyle f''} \\ I \otimes Y' & \xrightarrow[\gamma]{} & Y'' \end{array}$$

commutes. Let $X = (X', X''; \lambda)$ be a left $(R/J)_I$-module and let $X_I \subseteq X'$ be the R/J-submodule defined as

$$X_I = \{x' \in X' \mid \forall i \in I,\ \lambda(i \otimes x') = 0\}.$$

It is straightforward then to calculate the socle of X as $\mathrm{soc}(X) = (X_I, X''; 0)$. A left $(R/J)_I$-module X is called a *Grassmanian* if $X_I = 0$. The Grassmanians form an additive subcategory denoted by $\mathrm{Gr}(R/J, I) \subseteq (R/J)_I$-Mod. Since every left $(R/J)_I$-module X factors as a direct sum $X = X_0 \oplus (X_I, 0; 0)$ where X_0 is a Grassmanian, it is clear that every representation in $\mathrm{add}(R/J, 0; 0)$ is injective and that $\mathrm{Gr}(R/J, I) = (R/J)_I\text{-Mod} \mid \mathrm{add}(R/J, 0; 0)$.

The Grassmanian functor $\mathrm{Gr} : R\text{-Mod} \to \mathrm{Gr}(R/J, I)$ is defined as

$$\mathrm{Gr}(M) = (M/\mathrm{ann}_I(M), \mathrm{soc}(M); \alpha_M),$$

where $\alpha_M : {}_{R/J}I \otimes_{R/J} M/\mathrm{ann}_I(M) \to \mathrm{soc}(M)$ is given by $\alpha_M(i \otimes (m + \mathrm{ann}_I(M)) = im$. The following is due to Auslander.

Theorem 6.1 (see [2, Theorem 3.1, p. 64]). *Let $\mathcal{C}_I \subseteq R$-Mod be the additive subcategory of R-Mod whose objects are the left R-modules ${}_RM$ such that the annihilator $\mathrm{ann}_I(M)$ of I in M is an injective left R/I-module. The restriction $\mathrm{Gr}|_{\mathcal{C}_I}$ of the Grassmanian functor is a representation equivalence, that is, the functor $\mathrm{Gr} : \mathcal{C}_I \to \mathrm{Gr}(R/J, I)$ has the following properties:*

(1) *The functor $\mathrm{Gr}|_{\mathcal{C}_I}$ is dense, that is, for every $X \in \mathrm{Gr}(R/J, I)$, there is an ${}_RM \in \mathcal{C}_I \subseteq R$-Mod such that $\mathrm{Gr}(M) \cong X$.*

(2) *The functor $\mathrm{Gr}|_{\mathcal{C}_I}$ is full.*

(3) *If $f : {}_RM \to {}_RN$ is a morphism in $\mathcal{C}_I$ for which $\mathrm{Gr}(f) : \mathrm{Gr}(M) \to \mathrm{Gr}(N)$ is an isomorphism, then f is an isomorphism.* $\square$

Proposition 6.2. *Let R be a minimal counterexample to the Pure-semisimple Conjecture which is not right Morita. If I is the unique minimal ideal of R and J the Jacobson radical of R, then the formal lower triangular matrix ring $(R/J)_I$ is left pure-semisimple.*

Proof. Since the ring $(R/J)_I$ is left artinian, it is Krull–Schmidt. Towards application of Lemma 3.2, suppose that there is a non-noetherian sequence of morphisms $\{f_i : X_i \to X_{i+1}\}$ between finitely generated indecomposable left $(R/J)_I$-modules. If some X_n is not Grassmanian, then, since it is indecomposable, it must be a simple injective representation of the form $X_n = (S, 0; 0)$ where S is a simple left R/J-module. Because $f_k \neq 0$ for all $k \geq n$, this implies that for every $k \geq n$, $X_k \cong X_n$ and that every f_k is an isomorphism. From this contradiction, we infer that each X_i is Grassmanian and so by the first two properties stated in the theorem on Grassmanians, we may rewrite our non-noetherian sequence as $\{\mathrm{Gr}(g_i) \mid \mathrm{Gr}(M_i) \to \mathrm{Gr}(M_{i+1})\}$ where $\{g_i \mid M_i \to M_{i+1}\}$ is a sequence of morphisms between indecomposable, and hence finitely generated, left R-modules in $\mathcal{C}_I$. By the third property given in the theorem on Grassmanians, the sequence $\{g_i \mid M_i \to M_{i+1}\}$ is not noetherian, contradicting the hypothesis that R is left pure-semisimple. $\square$

In what follows, we will show that the ring $(R/J)_I$ is not right Morita and is therefore another counterexample to the Pure-semisimple Conjecture. If R is not right artinian, then, since R/I *is* right artinian, the ring $(R/J)_I$ cannot be right artinian and we are done. To prove the general case, we will apply the theorem on Grassmanians on the right side. So recall that a right $(R/J)_I$-module is of the form $Y = (Y', Y''; \rho)$ where Y' and Y'' are right R/J-modules and $\rho : Y'' \otimes_{R/J} I_{R/J} \to Y'_{R/J}$ is an R/J-morphism. The right Grassmanians are defined as those Y for which the socle $\mathrm{soc}(Y) = (Y', 0; 0)$.

Theorem 6.3. *If R is a minimal counterexample to the Pure-semisimple Conjecture and R is not right Morita, then the formal lower triangular matrix ring $(R/J)_I$ is yet another non-right Morita counterexample.*

Proof. We may assume, without loss of generality, that R, and hence $(R/J)_I$, is right artinian. There exists an indecomposable injective right R-module E_R which is not finitely generated. As R/I is of finite representation type, and therefore right Morita, the injective right R/I-module $\mathrm{ann}_I(E)$ is finitely generated. For a submodule M_R of E_R which contains $\mathrm{ann}_I(E) = \mathrm{ann}_I(M)$, consider the right Grassmanian $\mathrm{Gr}(M_R) = (\mathrm{soc}(M), M/\mathrm{ann}_I(M); \alpha_M) = (\mathrm{soc}(E), M/\mathrm{ann}_I(E); \alpha_M)$. It corresponds to a right $(R/J)_I$-module with simple socle $(\mathrm{soc}(E), 0; 0)$. As the length of M_R grows, so does that of $\mathrm{Gr}(M)$. The indecomposable injective right $(R/J)_I$-module $E(\mathrm{soc}(E), 0; 0)$ cannot therefore be finitely generated. $\square$

Write $R/J = \bigoplus_{i=1}^n R_i$ as a direct sum of simple artinian rings. As $I = \bigoplus_{i=1}^n R_i I$ is a minimal ideal, there is a unique i, say $i = 1$, such that $R_1 I \neq 0$ and in particular $R_1 I = I$. Similarly, there is a unique j such that $IR_j = I$. If $j \neq 1$, we will set $j = 2$. In either case, there is a decomposition

$$(R/J)_I = \begin{pmatrix} \bigoplus_{k\leq j} R_k & 0 \\ I & \bigoplus_{k\leq j} R_k \end{pmatrix} \oplus \bigoplus_{k>j} \begin{pmatrix} R_k & 0 \\ 0 & R_k \end{pmatrix},$$

where only the first factor is not semisimple artinian. This first factor, call it R', is therefore a counterexample to the Pure-semisimple Conjecture which is not right Morita.

Proposition 6.4. *Let $P_1, \ldots, P_m$ be a complete list (up to isomorphism), without repetitions, of the indecomposable projective left R'-modules. The basic ring*

$$\mathcal{B}(R') = \mathrm{End}_{R'}\Big(\bigoplus_{i=1}^{m} P_i\Big)$$

is a non-right Morita counterexample to the Pure-semisimple Conjecture. Furthermore, the Jacobson radical of $\mathcal{B}(R')$ is a minimal ideal.

Proof. As R' is semiperfect, it is Morita equivalent to its basic ring $\mathcal{B}(R')$ by [1, Proposition 27.14]. Hence the Jacobson radical $J(\mathcal{B}(R'))$ is a minimal ideal. By [1, Proposition 21.5(1)] for left modules, every left $\mathcal{B}(R')$-module is a direct sum of indecomposables and $\mathcal{B}(R')$ is therefore left pure-semisimple. That $\mathcal{B}(R')$ is not right Morita follows from [1, Propositions 21.6(2), 21.8(2) and (3)] for right modules. □

Let us calculate $\mathcal{B}(R')$ for the two cases $j = 1$ and $j = 2$. If $j = 1$, then

$$R' = \begin{pmatrix} R_1 & 0 \\ I & R_1 \end{pmatrix}$$

has only two indecomposable projective left R'-modules P_1 and P_2 and R' decomposes as

$$R' = (0, R_1; 0) \oplus (R_1, I; \alpha_R) = P_1^n \oplus P_2^n.$$

If we denote by F the division ring $(P_1, P_1) = (P_2, P_2)$, then

$$\mathcal{B}(R') = \begin{pmatrix} (P_1, P_1) & (P_2, P_1) \\ (P_1, P_2) & (P_2, P_2) \end{pmatrix} = \begin{pmatrix} F & 0 \\ (P_1, P_2) & F \end{pmatrix},$$

where (P_1, P_2) has an F–F-bimodule structure induced by $F = (P_2, P_2)$ on the left and $F = (P_1, P_1)$ on the right.

If $j = 2$, then we have $R_1 I R_2 = I$ and

$$R' = \begin{pmatrix} R_1 \oplus R_2 & 0 \\ I & R_1 \oplus R_2 \end{pmatrix}.$$

As a left R'-module R' decomposes into projective indecomposables as

$$R' = P_1^m \oplus P_2^n \oplus P_3^m \oplus P_4^n$$

where

$$P_1^m = (0, R_1; 0), \quad P_2^n = (R_2, I; \alpha_R), \quad P_3^m = (R_1, 0; 0), \quad P_4^n = (0, R_2; 0).$$

We have the division rings $F = (P_1, P_1) = (P_3, P_3)$ and $G = (P_2, P_2) = (P_4, P_4)$ and the only nontrivial Hom set between distinct projective indecomposables is (P_1, P_2). Therefore,

$$\begin{aligned} \mathcal{B}(R') &= \begin{pmatrix} (P_1, P_1) & 0 \\ (P_1, P_2) & (P_2, P_2) \end{pmatrix} \oplus F \oplus G \\ &= \begin{pmatrix} F & 0 \\ (P_1, P_2) & G \end{pmatrix} \oplus F \oplus G. \end{aligned} \tag{4}$$

Theorem 6.5. *There is an enumeration $P_1, \ldots, P_{2j}$ of indecomposable projective left R'-modules such that the only nontrivial* Hom *set between distinct projective indecomposables is* $\mathrm{Hom}_{R'}(P_1, P_2)$. *The endomorphism ring of each P_i is a* *division ring. Let* ${}_GB_F = J(R')$ *denote the simple G–F-bimodule* (P_1, P_2) *where* $F = (P_1, P_1)$ *and* $G = (P_2, P_2)$. *The lower triangular matrix ring*

$$R_B = \begin{pmatrix} F & 0 \\ B & G \end{pmatrix}$$

is a non-right Morita counterexample to the Pure-semisimple Conjecture.

Proof. The first two statements have been verified above. When $j = 1$, the ring R' is already of the sought form with $F = G$. When $j = 2$, the last statement follows from (4). □

The indecomposable injective left R_B-modules are $E_1 = ({}_F(B,G)_G, {}_GG; \mathrm{Ev})$ where $\mathrm{Ev} : {}_GB \otimes_F (B,G) \to {}_GG$ is given by $\mathrm{Ev}(b \otimes \mu) = \mu(b)$ and $E_2 = ({}_FF, 0; 0)$. The respective endomorphism rings are $(E_1, E_1) = G^{\mathrm{op}}$ and $(E_2, E_2) = F^{\mathrm{op}}$ and it is immediate that $(E_1, E_2) = {}_G((B,G),F)_F$ and $(E_2, E_1) = 0$. Since R_B is left pure-semisimple it is left Morita with a Morita duality $\mathcal{D} : R_B\text{-mod} \to \text{mod-}S$ where

$$S = \mathrm{End}_{R_B}(E_1 \oplus E_2) = \begin{pmatrix} G^{\mathrm{op}} & 0 \\ {}_G((B,G),F)_F & F^{\mathrm{op}} \end{pmatrix}.$$

By Theorem 3.3, the ring S is left pure-semisimple and hence left artinian. Because S is right Morita, it is right artinian. Thus the G–F-bimodule ${}_GB'_F = ((B,G),F)$ is finite-dimensional both as F-vector space and as G-vector space.

Proposition 6.6. *Let F and G be division rings, each of which is finite-dimensional as a vector space over its respective center, and let* ${}_GB_F$ *be a G–F-bimodule. If the formal lower triangular matrix ring*

$$R_B = \begin{pmatrix} F & 0 \\ B & G \end{pmatrix}$$

is left pure-semisimple, then it is right Morita.

Proof. Dowbor and Simson [9, Proposition 1.3] proved that if F and G are as given and ${}_GB'_F$ is a G–F-bimodule finite-dimensional both as F-vector space and as G-vector space, then the same is true of the F–G-bimodule ${}_F(B',F)_G$. From the discussion preceding this proposition, we may apply this result to the G–F-bimodule $B' = {}_G((B,G),F)_F$, proving that the F–G-bimodule ${}_F(B,G)_G$ is also finite-dimensional on both sides. Another application of the same shows that ${}_GB_F$ is finite dimensional on both sides and hence that R_B is right artinian.

The injective right R_B-modules are $(0, G; 0)$, which is simple, and $(F_F, (B,F)_G; \mathrm{Ev})$. Yet another application of the result by Dowbor and Simson shows that $(B,F)_G$ is a finite-dimensional right F-vector space. Both of these injectives are then finitely generated and R_B is therefore right Morita. □

Starting from an arbitrary counterexample R, we have arrived at the counterexample of Theorem 6.5 by performing three types of operations. Namely, we have applied Morita duality, Morita equivalence and we have, on more than one occasion, modded out by an ideal. It follows that each of the remaining division rings F and G is nothing more than the endomorphism ring of some simple R-module over the original ring R. If R is a polynomial identity ring, then by Kaplansky's Theorem (cf. [18, Theorem 1.1, p. 37]), both F and G are finite-dimensional vector spaces over their respective centers.

Theorem 6.7. *Every left pure-semisimple polynomial identity ring is of finite representation type.* □

We noticed above how the ring R_B fails to be right Morita iff one of the right vector spaces B_F and $(B,F)_G$ is infinite dimensional. Suppose that R_B is a non-right Morita counterexample to the Pure-semisimple conjecture. If R_B is right artinian, then $(B,F)_G$ is not finite-dimensional. We will prove that, in that case, the formal lower triangular matrix ring $R_{(B,F)}$ is yet another counterexample to the Pure-semisimple Conjecture, one that is not right artinian.

Proposition 6.8. *Let F and G be division rings and ${}_GB_F$ a G–F-bimodule such that the ring R_B is a right artinian, but not right Morita counterexample to the Pure-semisimple Conjecture. Then the formal lower triangular matrix ring*

$$R_{(B,F)} = \begin{pmatrix} G & 0 \\ (B,F) & F \end{pmatrix}$$

is a non-right artinian counterexample to the Pure-semisimple Conjecture.

Proof. First note that $\dim_F(B,F) = \dim B_F$ is finite so that $R_{(B,F)}$ is left artinian. We will apply Lemma 3.2 to show that $R_{(B,F)}$ is left pure-semisimple; note that a left $R_{(B,F)}$-module is finitely presented iff it is finitely generated. By the use of reflection functors [7,19], we will indicate how the categories $R_B\text{-mod} \mid \operatorname{add}(0,G;0)$ and $R_{(B,F)}\text{-mod} \mid \operatorname{add}(G,0;0)$ are equivalent. The result then follows immediately.

Let $X = ({}_FX', {}_GX''; \lambda) \in R_B\text{-mod} \mid \operatorname{add}(0,G;0)$. This is equivalent to the G-morphism λ being an epimorphism. Let us define the reflection functor

$$C^+ : R_B\text{-mod} \mid \operatorname{add}(0,G;0) \to R_{(B,F)}\text{-mod} \mid \operatorname{add}(G,0;0)$$

at X by considering λ as part of a short exact sequence

$$0 \to {}_G\operatorname{Ker}\lambda \xrightarrow{\ker\lambda} {}_GB \otimes_F X' \xrightarrow{\lambda} X'' \to 0.$$

As ${}_FX'$ and ${}_GB$ are finite-dimensional, so are ${}_GB \otimes_F X'$ and ${}_G\operatorname{Ker}\lambda$. As B_F is finite-dimensional, there is an isomorphism

$$_GB \otimes_F X' \cong {}_G({}_F(B,F)_G, {}_FX')$$

natural for finite-dimensional $_FX'$. The image of $\ker\lambda$ under the natural isomorphism

$$\begin{aligned}({}_G\operatorname{Ker}\lambda, {}_GB \otimes_F X') &\cong ({}_G\operatorname{Ker}\lambda, {}_G({}_F(B,F), {}_FX')) \\ &\cong ({}_F(B,F) \otimes_G \operatorname{Ker}\lambda, {}_FX')\end{aligned}$$

is an F-monomorphism denoted by $(\ker\lambda)' : {}_F(B,F) \otimes_G \operatorname{Ker}\lambda \to {}_FX'$. Define

$$C^+(X) = ({}_G\operatorname{Ker}\lambda, {}_FX; (\ker\lambda)') \in R_{(B,F)}\text{-mod} \mid \operatorname{add}(G,0;0).$$

The functor $C^- : R_{(B,F)}\text{-mod} \mid \operatorname{add}(G,0;0) \to R_B\text{-mod} \mid \operatorname{add}(0,G;0)$ is defined for $Y = ({}_GY', {}_FY''; \gamma) \in R_{(B,F)}\text{-mod} \mid \operatorname{add}(G,0;0)$ by considering the cokernel of the F-monomorphism $\gamma : {}_F(B,F) \otimes_G Y' \to {}_FY''$ and retracing the definition of C^+. The pair (C^+, C^-) then constitutes an equivalence of categories [7,19]. □

Theorem 6.9. *If there is a counterexample to the Pure-semisimple Conjecture, then there exists a counterexample R which is (left and right) hereditary, not right artinian and whose Jacobson radical $J(R)$ is a minimal ideal.* □

Let us check how some of the general results regarding left pure-semisimple rings apply to a counterexample R_B where $_GB_F$ is a simple G–F-bimodule infinite-dimensional as an F-vector space. If $V_F \leq B_F$ is a finite-dimensional F-subspace of B_F, the triple $((B/V)_F, G_G; \pi_V)$ where $\pi_V : G \otimes_G B_F \to (B/V)_F$ is the natural quotient map, corresponds to a finitely presented indecomposable right R_B-module. The endomorphism ring is easily computed as

$$G(V) = \{g \in G \mid gV \leq V\}.$$

This is a division ring contained in G and by Proposition 3.6, G is finite-dimensional as a left vector space over $G(V)$. However, as a right vector space over $G(V)$, G cannot be finite-dimensional. For if $G = \sum_{i=1}^n g_iG(V)$, then the G–F-subbimodule

$$GV_F = \sum_{i=1}^n g_iG(V)V_F = \sum_{i=1}^n g_iV_F$$

of $_GB_F$ would be finite-dimensional over F, contradicting the simplicity of $_GB_F$.

Fix a natural number n. The multiplicative group $G^\times$ of nonzero elements acts on the set of n-dimensional F-subspaces of B_F. Two such vector spaces V_F and W_F lie in the same $G^\times$-orbit iff the finitely presented indecomposable right R_B-modules $((B/V)_F, G_G, \pi_V)$ and $((B/W)_F, G_G, \pi_W)$ are isomorphic. By [17, Section 8.4], [28] or [24], it is known that there are only finitely many isomorphism types and hence just finitely many $G^{\times}$-orbits of n-dimensional F-subspaces of B_F.

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