FINITELY PRESENTED RIGHT MODULES OVER A LEFT PURE-SEMISIMPLE RING

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In this note, we study rings over which every left module is a direct sum of finitely generated modules. Chase [5] showed that all such rings are left artinian and Fuller [8] proved that these are precisely the rings R with the property that every left R-module ${}_RM$ has a decomposition ${}_RM = \oplus_{\iota\in I} U_\iota$ which complements direct summands. This means that whenever ${}_RK$ is a direct summand of ${}_RM$, there is a subset J of the index set I such that ${}_RM = {}_RK \oplus {}_R(\oplus_{\iota\in J} U_\iota)$. Zimmermann-Huisgen [22, Corollary 2] showed that these are the *left pure-semisimple rings* defined below. And although left pure-semisimple rings have many attractive features (see [15, p. 210]), it is not known whether they are all of *finite representation type*, that is, left artinian with only finitely many indecomposable left modules.

Auslander [1] showed that every left pure-semisimple *artin algebra* is of finite representation type (an artin algebra is a ring which is finitely generated as a module over an artinian centre). If Λ is an artin algebra and M_Λ is a right Λ-module, then M is also finitely generated as a module over the centre of Λ. Consequently, M_Λ is of finite length as a module over its endomorphism ring $\mathrm{End}_\Lambda M$; a module with this property is called *endofinite*. We observe below how, at least in this respect, left pure-semisimple rings resemble artin algebras.

THEOREM 2.3. *Every finitely presented right R-module over a left pure-semisimple ring R is endofinite.*

This is an extension of Chase's result which may be thought of as the same statement restricted to the finitely generated projective right modules.

A short exact sequence of left R-modules

$$0 \longrightarrow {}_RM \xrightarrow{f} {}_RN \xrightarrow{g} {}_RK \longrightarrow 0 \tag{1}$$

is called *pure-exact* if it satisfies one of the following equivalent conditions [18, Proposition 3].

- For each (finitely presented) right R-module X_R, the sequence of abelian groups

$$0 \longrightarrow X \otimes_R M \xrightarrow{X\otimes f} X \otimes_R N \xrightarrow{X\otimes g} X \otimes_R K \longrightarrow 0$$

is exact.

- For every finitely presented left R-module ${}_RY$, the sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(Y,M) \xrightarrow{\mathrm{Hom}(Y,f)} \mathrm{Hom}_R(Y,N) \xrightarrow{\mathrm{Hom}(Y,g)} \mathrm{Hom}_R(Y,K) \longrightarrow 0$$

is exact.

Received 30 July 1992; revised 26 May 1993.

1991 *Mathematics Subject Classification* 16B70, 16D70, 16G60.

The author is a National Science Foundation Postdoctoral Fellow.

Note that the first (respectively second) condition is really a condition on the morphism f (respectively g); a morphism with this property is called a *pure-monomorphism* (respectively *pure-epimorphism*). A left R-module ${}_RM$ (respectively ${}_RK$) is called *pure-injective* (respectively *pure-projective*) if every pure-exact sequence of the form (1) is split-exact.

The ring R is called *left pure-semisimple* if every pure-exact sequence of left R-modules is split-exact. This is equivalent to the property that every left R-module is pure-injective (respectively pure-projective). A left R-module ${}_RM$ is called *Σ-pure-injective* if for every index set I, the direct sum ${}_RM^{(I)}$ of I copies of ${}_RM$ is again pure-injective. Evidently, every left module over a left pure-semisimple ring is Σ-pure-injective.

The condition that a ring R be of finite representation type is left-right symmetric [7, Theorem 1.2], and it is shown in [**17**, **9**, **22**] that a ring is of finite representation type if and only if it is left and right pure-semisimple. So to prove that a left pure-semisimple ring is of finite representation type it is enough to show it is right pure-semisimple. Since every endofinite module is Σ-pure-injective (see Proposition 1.1 below), Theorem 2.3 takes the following step in that direction.

COROLLARY 2.4. *Every finitely presented right R-module over a left pure-semisimple ring R is Σ-pure-injective.*

1. *Preliminaries*

Let R be a ring, associative and with a unit $1 \in R$. Recall from [**14**, §2.1] the notion of a *positive-primitive formula*, or *pp-formula* (in one variable), in the language of right R-modules. This is nothing more than an existentially quantified system of linear equations with coefficients in R. More precisely, let $C = (c_{ij})_{i=0,j=1}^{m;n}$ be an $(m+1)\times n$ matrix with entries in R. Consider the system of linear equations whose matrix of coefficients is C:

$$xc_{0j} + \sum_{i=1}^{m} y_i c_{ij} \doteq 0, \quad 1 \leqslant j \leqslant n.$$

If we write the matrix $C = \begin{pmatrix} C_0 \\ C' \end{pmatrix}$, where C_0 is the first row of C and C' is the remaining $m \times n$ submatrix, we can express this system by the single equation

$$(x, \mathbf{y})\begin{pmatrix} C_0 \\ C' \end{pmatrix} \doteq \mathbf{0},$$

where $\mathbf{y} = (y_1, \dots, y_m)$. Existentially quantifying over the variables $\mathbf{y}$ gives a formula $\phi(x)$ in the free variable x

$$\exists \mathbf{y}(x, \mathbf{y})\begin{pmatrix} C_0 \\ C' \end{pmatrix} \doteq \mathbf{0}; \tag{2}$$

this is the general form of a pp-formula in one variable for right R-modules.

Given a right R-module M_R, the pp-formula $\phi(x)$ defines a subgroup $\phi(M)$ of M defined by

$$\phi(M) := \left\{ a \in M \colon \text{there is a } \mathbf{b} \in M^m \text{ such that } (a, \mathbf{b})\begin{pmatrix} C_0 \\ C' \end{pmatrix} = \mathbf{0} \right\}.$$

This is, in some sense, the subgroup of solutions to $\phi(x)$ in M_R. A subgroup of M defined in this way is called a *finite matrix subgroup*, or *pp-definable subgroup*, of M_R. Finite matrix subgroups were introduced by Gruson and Jensen [**11**] and Zimmermann [**20**]. As pp-definable subgroups, they have been studied by Baur [**4**] and Garavaglia [**10**]. Two pp-formulae $\phi(x)$ and $\psi(x)$ are said to be equivalent if for every right R-module N_R, they define the same finite matrix subgroup, $\phi(N) = \psi(N)$.

Positive-primitive formulae are defined similarly for left R-modules. Our need for these formulae stems from the following characterization of Σ-pure-injective modules.

PROPOSITION 1.1 [**12**; **20**, Folgerung 3.4]. *A left R-module $_RM$ is Σ-pure-injective if and only if it satisfies the descending chain condition (dcc) on finite matrix subgroups. Thus a ring R is left pure-semisimple if and only if every left R-module has the dcc on finite matrix subgroups.*

Prest [**13**; **14**, §8.4] noticed that there is a bijective correspondence between the right pp-formulae and the left pp-formulae (modulo equivalence). For a right pp-formula of the form (2), define $\phi^*(x)$ to be the pp-formula for left R-modules

$$\exists \mathbf{z} \begin{pmatrix} 1 & C_0 \\ \mathbf{0} & C' \end{pmatrix} \begin{pmatrix} x \\ \mathbf{z} \end{pmatrix} \doteq \mathbf{0},$$

where $\mathbf{z} = (z_1, \ldots, z_n)$ and the $\mathbf{0}$ below the 1 is a column vector with m entries. That this is well-defined can be shown using the following result of Zimmermann-Huisgen and Zimmermann.

PROPOSITION 1.2 [**23**, Proposition 3]. *Let M_R be a right R-module and let S be a ring such that there is an S-R-bimodule structure $_SM_R$. Let $_SW$ be an injective cogenerator. The group $M^* = \mathrm{Hom}_S(_SM_R, {_SW})$ has a left R-module structure given by $(rf)(a) = f(ar)$. If $\phi(x)$ is a right pp-formula, the finite matrix subgroup $\phi(M_R)$ of M is an S-submodule and*

$$\phi^*(_RM^*) = \mathrm{Hom}_S(M/\phi(M), {_SW}).$$

The function $\phi(M_R) \mapsto \phi^*(_RM^*)$ is a bijective correspondence between the finite matrix subgroups of M_R and those of $_RM^*$.

In general, a finite matrix subgroup $\phi(M_R)$ of M need not be an R-submodule of M_R. But if it happens to be such, it is clear by Proposition 1.2 that the finite matrix subgroup $\phi^*(_RM^*)$ is then an R-submodule of the left R-module $_RM^*$. Proposition 1.2 may also be used to verify the following.

PROPOSITION 1.3 [**13**, Proposition 1.3; **23**, Theorem 6]. *A ring R is left pure-semisimple if and only if every right R-module satisfies the ascending chain condition on finite matrix subgroups.*

2. *Finitely presented indecomposables*

Since every left pure-semisimple ring R is left artinian, the *Auslander–Bridger transpose* [**2**] may be used to show that the finitely presented *right* R-modules are also well-behaved. When this is done as in [**19**, §2], it shows that every finitely presented right R-module is a direct sum of finitely many indecomposables each of which has a local endomorphism ring. To prove our main result, that every finitely presented right module over a left pure-semisimple ring is endofinite, it is enough to verify it for indecomposable modules. So even if some of our statements hold for finitely presented modules in general, we shall restrict them to the indecomposable case for ease of argument.

Let A_R be a finitely presented R-module with a local endomorphism ring $S = \operatorname{End}_R A_R$; so there is an S-R-bimodule structure ${}_SA_R$. Let ${}_SV$ be the *minimal injective cogenerator* in the category of left S-modules. This just means that ${}_SV$ is the injective envelope of the unique simple left S-module. Consider the left R-module ${}_RA^\# = \operatorname{Hom}_S({}_SA_R, {}_SV)$. By [**16**, Exercise I.33], we have that ${}_SA \otimes_R A^\# \cong {}_SV$. Let $T = \operatorname{End}_R A^\#$ be the endomorphism ring of ${}_RA^\#$. Then

$$T = \operatorname{Hom}_R({}_RA^\#, {}_R\operatorname{Hom}_S({}_SA_R, {}_SV)) = \operatorname{Hom}_S({}_SA \otimes_R A^\#, {}_SV) = \operatorname{End}_S V.$$

The ring T, being the endomorphism ring of the indecomposable injective module ${}_SV$, must be local, so that ${}_RA^\#$ is an indecomposable left R-module. If R is left pure-semisimple, then ${}_RA^\#$ must be finitely generated. We shall also need the following observation of Zimmermann-Huisgen and Zimmermann.

PROPOSITION 2.1 [**23**, Observation 8]. *Let F_R be a finitely presented right R-module with the ascending chain condition on finite matrix subgroups, and let $S = \operatorname{End}_R F$. A subgroup of F is an S-submodule of ${}_SF$ if and only if it is a finite matrix subgroup of F_R. In particular, ${}_SF$ is noetherian.*

PROPOSITION 2.2. *If R is left pure-semisimple and A_R is a finitely presented indecomposable right R-module, then A_R is endofinite.*

Proof. Since $S = \operatorname{End}_R A$ is local and ${}_SA$ is noetherian, there is a descending chain of S-submodules

$${}_SA \supseteq \operatorname{rad}_S A \supseteq \operatorname{rad}_S^2 A \supseteq \ldots \supseteq \operatorname{rad}_S^n A \supseteq \ldots$$

where every nonzero $\operatorname{rad}_S^i A$ contains $\operatorname{rad}_S^{i+1} A$ properly. Since every factor of this filtration is semisimple of finite length, all we need to show is that there is a k such that $\operatorname{rad}_S^k A = 0$. By Proposition 2.1, there is a pp-formula $\phi_i(x)$ such that $\phi_i(A) = \operatorname{rad}_S^i A$. Note that each $\phi_i(A)$ is an S-R-subbimodule of ${}_SA_R$. If the above chain were infinite, we would obtain, by Proposition 1.2, a strictly ascending chain

$$\phi_0^*(A^\#) \subset \phi_1^*(A^\#) \subset \ldots \subset \phi_n^*(A^\#) \subset \ldots \subset {}_RA^\#$$

of submodules of ${}_RA^\#$. But the left R-module ${}_RA^\#$ is finitely generated and so must be of finite length. This is a contradiction.

Now a finite direct sum of modules is endofinite if and only if each of its factors is endofinite.

THEOREM 2.3. *Every finitely presented right R-module over a left pure-semisimple ring R is endofinite.*

In conjunction with Proposition 1.1, Theorem 2.3 yields the following.

COROLLARY 2.4. *Every finitely presented right R-module over a left pure-semisimple ring R is Σ-pure-injective.*

EXAMPLE. Let F and G be division rings and ${}_GB_F$ a G-F-bimodule. Consider the matrix ring $R = \begin{pmatrix} F & 0 \\ {}_GB_F & G \end{pmatrix}$ consisting of the matrices $\begin{pmatrix} f & 0 \\ b & g \end{pmatrix}$ where $f \in F$, $b \in B$ and $g \in G$. It is easy to check that R is left artinian if and only if the dimension of the left G-vector space ${}_GB$ is finite and that R is right artinian if and only if the right F-vector space B_F is finite dimensional. Simson [**15**, Theorem 3.3] has shown that if there is a hereditary example of a ring which is left pure-semisimple but not of finite representation type, then there is an example of the form $R = \begin{pmatrix} F & 0 \\ {}_GB_F & G \end{pmatrix}$ which, in addition, fails to be right artinian.

Suppose $R = \begin{pmatrix} F & 0 \\ {}_GB_F & G \end{pmatrix}$ were such a counterexample, and let us see what Theorem 2.3 says regarding the finitely presented local right R-modules. (A local module is a module with a unique maximal proper submodule.) By [**16**, Exercises I.10, I.35], the right R-modules may be represented as triples $(X_F, Y_G, \alpha\colon Y \otimes {}_GB_F \to X_F)$, where X_F is a right F-vector space, Y_G is a right G-vector space and α is an F-linear map. The indecomposable projective right R-modules are $P_1 = e_1 R = (F, 0, 0)$ and $P_2 = e_2 R = ({}_GB_F, G_G, \beta\colon G \otimes {}_GB_F \to B_F)$, where β is the canonical isomorphism. Now $\mathrm{End}_R P_2 = e_2 Re_2 = G$, so to say that P_2 is endofinite just means that the dimension of ${}_GB$ is finite.

The finitely generated proper submodules of P_2 are the modules $(B'_F, 0, 0)$ where B'_F is a finite dimensional F-subspace of B_F. The finitely presented quotients of P_2 are thus of the form $((B/B')_F, G_G, \pi\colon G \otimes {}_GB_F \to (B/B')_F)$, where π is the composition of β above with the natural quotient map. The endomorphism ring $\mathrm{End}_R((B/B')_F, G_G, \pi)$ $= G' = \{g \in G\colon g(B') \subseteq B'\}$ is again a division ring. Because B'_F is finite dimensional, $g(B') \subseteq B'$ implies $g(B') = B'$ when $g \neq 0$. Now $((B/B')_F, G_G, \pi)$ is finite dimensional over G' if and only if both of the left G'-vector spaces ${}_{G'}(B/B')$ and ${}_{G'}G$ are finite dimensional. But if ${}_{G'}G$ is finite dimensional, then since ${}_GB$ is finite dimensional, we already have that ${}_{G'}(B/B')$ is finite dimensional. Thus a necessary condition that R be a counterexample is that for every finite dimensional F-subspace B'_F of B_F, the division ring G is finite dimensional as a left vector space over the division ring $G' = \{g \in G\colon g(B') \subseteq B'\}$.

One of the main tools used in the study of modules over an artin algebra Λ is the Morita duality which exists between the finitely generated left Λ-modules and the finitely generated right Λ-modules. The next result further attests to the resemblance, in this regard, of a left pure-semisimple ring to an artin algebra.

COROLLARY 2.5. *Let R be left pure-semisimple and A_R a finitely presented indecomposable right R-module. Let $S = \mathrm{End}_R A_R$ be the endomorphism ring of A_R, ${}_SV$ the minimal injective cogenerator in the category of left S-modules, and let ${}_RA^\sharp =$ $\mathrm{Hom}_S({}_SA_R, {}_SV)$. If $T^{\mathrm{op}} = \mathrm{End}_S V = \mathrm{End}_R A^\sharp$, then the covariant functor $D(-) =$ $\mathrm{Hom}_S(-, {}_SV_T)$ constitutes a Morita duality $D\colon S\text{-mod} \to \text{mod-}T$ between the category of finitely generated left S-modules and that of finitely generated right T-modules. Furthermore, $D({}_SA) = A^\sharp{}_T$.*

Proof. It is enough to show that S is left artinian and that ${}_SV$ is a finitely generated S-module; by [**3**], the contravariant functor $\mathrm{Hom}_S(-, {}_SV_T)$ is then a Morita duality. Let $a_1, \ldots, a_n \in A_R$ be a sequence of generators for A_R as an R-module. The

n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ is an element of the finite length S-module ${}_SA^n$. The cyclic submodule $S\mathbf{a}$ is faithful, so that S must be left artinian. To see that ${}_SV \cong {}_SA \otimes {}_RA^\#$ is finitely generated, just use the fact that ${}_RA^\#$ and ${}_SA$ are finitely generated.

Corollary 2.5 asserts that when R is a left pure-semisimple ring, there is an 'endoduality' between the finitely presented indecomposable right R-modules A_R and the left R-modules ${}_RA^\#$. By [**6**, Theorem 2.3], the modules ${}_RA^\#$ are precisely the indecomposable left R-modules which are the domain of a *left almost split morphism*. Indeed, by [**21**, Satz 1], if $A^\#$ is not injective, there is an *almost split sequence* of left R-modules

$$0 \longrightarrow A^\# \longrightarrow X \longrightarrow \mathrm{Tr}(A) \longrightarrow 0,$$

where $\mathrm{Tr}(A)$ denotes the Auslander–Bridger transpose of A_R.

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