The Recollements of Purity

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Outline

Historical definition

Prüfer: Let N be an abelian group. A subgroup $M \leq N$ is pure in N if it is relatively divisible in N: for every $n \in \mathbb{Z}$,

$$M \cap nN = nM.$$

Definition (Fieldhouse)

Let R be an associative ring with 1. A left R-submodule ${}_RM \leq {}_RN$ is pure in ${}_RN$ if for every $m \times n$ matrix A with entries in R,

$$M^m \cap AN^n = AM^n.$$

A short exact sequence in $\mathrm{R\text{-}Mod}$,

$$0 \longrightarrow {}_RM \stackrel{i}{\longrightarrow} {}_RN \longrightarrow {}_RK \longrightarrow 0$$

is pure exact if $\mathrm{Im}\, i \leq N$ is a pure submodule.

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Modern definitions

A short exact sequence in R-Mod,

$$0 \longrightarrow {}_R M \longrightarrow {}_R N \longrightarrow {}_R K \longrightarrow 0$$

is pure exact iff:

- **Cohn:** for every finitely presented right R-module X_R, the sequence of abelian groups

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  is exact; iff

- **Warfield:** for every finitely presented left R-module ${}_R Y$, the sequence of abelian groups

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  is exact. **Abbreviation:** $(Y, -) := \mathrm{Hom}_R(Y, -)$.

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Fragments of exactness

Let $\mathcal{A}$ be a small abelian category.

$(\mathcal{A}, \mathrm{Ab})$ = the category of additive functors $F : \mathcal{A} \to \mathrm{Ab}$.

Important subcategories of $(\mathcal{A}, \mathrm{Ab})$

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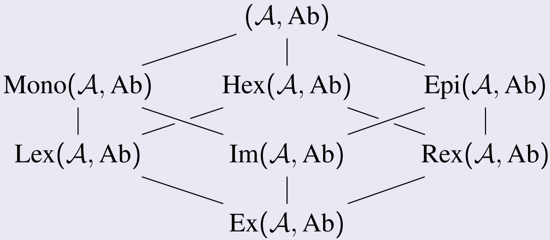
Important subcategories of $(\mathcal{A}, \mathrm{Ab})$



Coherent functors

$\mathrm{R\text{-}mod}$ = the category of finitely presented left R-modules.

Definition

A functor $F \in (\mathrm{R\text{-}mod}, \mathrm{Ab})$ is finitely presented, or coherent, if there exists a presentation

$$(Y, -) \xrightarrow{(f,-)} (X, -) \longrightarrow F \longrightarrow 0$$

in $(\mathrm{R\text{-}mod}, \mathrm{Ab})$ by representable objects, $f : X \to Y$ in $\mathrm{R\text{-}mod}$.

Example: The forgetful functor $(R, -) \cong R \otimes_R -$ is coherent.

Theorem (Auslander)

The subcategory $\mathrm{fp}(\mathrm{R\text{-}mod}, \mathrm{Ab}) \subseteq (\mathrm{R\text{-}mod}, \mathrm{Ab})$ is abelian.

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The resolution/coresolution of a matrix A

Let $X_R \in \mathrm{mod\text{-}R}$ be the module presented by the matrix A,

$$R^n_R \xrightarrow{A\times -} R^m_R \longrightarrow X_R \longrightarrow 0.$$

Definition: The resolution/coresolution of A

is the diagram in $(\mathrm{mod\text{-}R}, \mathrm{Ab})$ with exact rows

$$\begin{array}{ccccccccc} & & R^n \otimes_R - & \xrightarrow{A\times -} & R^m \otimes_R - & \longrightarrow & X \otimes_R - & \longrightarrow & 0 \\ & & \| & & \| & & & & \\ 0 \longrightarrow (\mathrm{Tr}\, X, -) & \longrightarrow & ({}_R R^n, -) & \xrightarrow{A\times -} & ({}_R R^m, -), & & & & \end{array}$$

where $\mathrm{Tr}\, X$ denotes the **Auslander-Bridger Transpose.**

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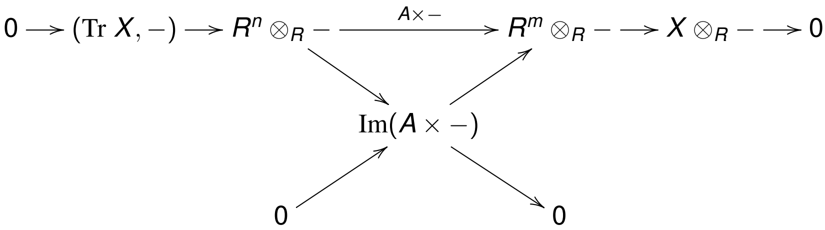
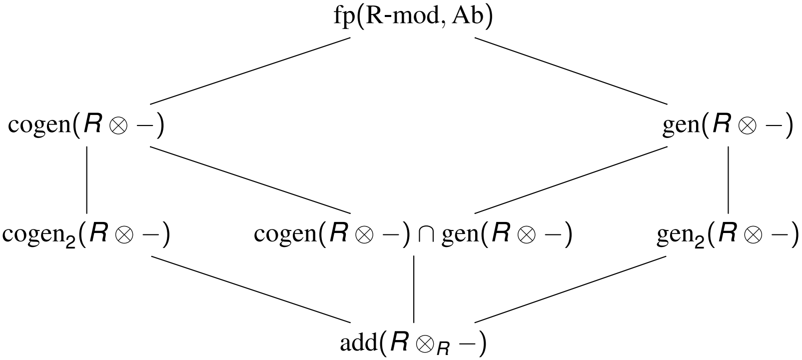
The equivalent definitions of purity



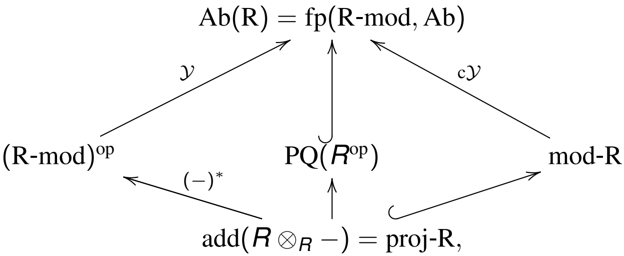
The Snake Lemma:

If $\Sigma : 0 \longrightarrow {}_RM \longrightarrow {}_RN \longrightarrow {}_RK \longrightarrow 0$ is a short exact sequence in $\mathrm{R\text{-}Mod}$, then $(\mathrm{Tr}X, \Sigma)$ is exact iff $\mathrm{Im}(A \times \Sigma)$ is exact iff $X \otimes_R \Sigma$ is exact.

The important subcategories of $\mathrm{fp}(\text{R-mod}, \mathrm{Ab})$



The subcategories that test for purity



Freyd: $\mathrm{Ab}(\mathrm{R})$ is the free abelian category over $\mathrm{add}(R \otimes -)$.

Crawley-Boevey & Sauter: The Projective Quotient Category

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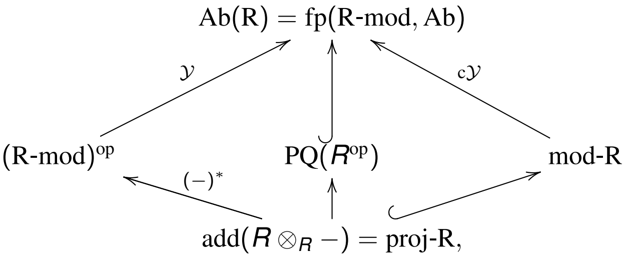

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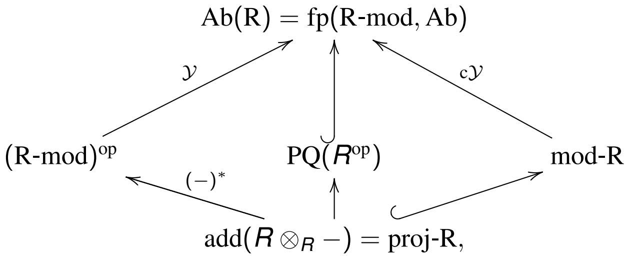

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Outline

Definition (Beilinson, Bernstein & Deligne)

A recollement of abelian categories is a diagram of functors between abelian categories of the form

$$R(\mathcal{W},\mathcal{X},\mathcal{Z}):\quad \mathcal{W} \underset{\tau}{\overset{\varphi}{\leftleftarrows}} \xrightarrow{\iota} \mathcal{X} \underset{r}{\overset{\ell}{\leftleftarrows}} \xrightarrow{p} \mathcal{Z}$$

satisfying the following properties:

1. (φ, ι, τ) and (ℓ, p, r) are **adjoint triples;**
2. the functors ι, ℓ and r are fully faithful;
3. $\mathrm{Im}\,\iota = \mathrm{Ker}\ p$.

The left adjoint $\ell : \mathcal{Z} \to \mathcal{X}$ is right exact; the right adjoint $r : \mathcal{Z} \to \mathcal{X}$ is left exact; and the middle adjoint $m : \mathcal{Z} \to \mathcal{X}$, given by the image of the norm, preserves images.

The stable functor category

Let $\mathcal{B} \subseteq \mathcal{A}$ be an inclusion of small additive categories and $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ the quotient category:

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(\underline{A}, \underline{B}) = \mathrm{Hom}_{\mathcal{A}}(A, B)/\mathcal{B}(A, B),$$

the subgroup of morphisms modulo those that factor through an object in $\mathcal{B}$.

Theorem

Precomposition with π induces a recollement of abelian categories

$$(\mathcal{A}/\mathcal{B}, \mathrm{Ab}) \xrightarrow{\pi^*} (\mathcal{A}, \mathrm{Ab}) \xrightarrow{\mathrm{res}_{\mathcal{B}}} (\mathcal{B}, \mathrm{Ab}),$$

where $\mathrm{res}_{\mathcal{B}} : (\mathcal{A}, \mathrm{Ab}) \to (\mathcal{B}, \mathrm{Ab})$ is the restriction functor and $(\mathcal{A}/\mathcal{B}, \mathrm{Ab}) \subseteq (\mathcal{A}, \mathrm{Ab})$ is the subcategory of functors that vanish on $\mathcal{B}$.

The Eilenberg-Watts Theorem

1. $\mathrm{R\text{-}Mod} \to (\mathrm{mod\text{-}R}, \mathrm{Ab})$, $M \mapsto \mathrm{Ev(M)}|_{\mathrm{mod\text{-}R}} = - \otimes_{\mathrm{R}} \mathrm{M}$ is a fully faithful embedding of $\mathrm{R\text{-}Mod}$ onto $\mathrm{rex}(\mathrm{mod\text{-}R}, \mathrm{Ab})$;
2. $\mathrm{R\text{-}Mod} \to (\mathrm{mod\text{-}R}, \mathrm{Ab})$, $M \mapsto \mathrm{Ev(M)}|_{(\mathrm{R\text{-}mod})^{\mathrm{op}}} = \mathrm{Hom}_{\mathrm{R}}(-, \mathrm{M})$ is a fully faithful embedding of $\mathrm{R\text{-}Mod}$ onto $\mathrm{lex}((\mathrm{R\text{-}mod})^{\mathrm{op}}, \mathrm{Ab})$;
3. $\mathrm{R\text{-}Mod} \to (\mathrm{PQ}(\mathrm{R}^{\mathrm{op}}), \mathrm{Ab})$, $M \mapsto \mathrm{Ev(M)}|_{\mathrm{PQ}(\mathrm{R}^{\mathrm{op}})}$ is a fully faithful embedding of $\mathrm{R\text{-}Mod}$ onto $\mathrm{im}(\mathrm{PQ}(R^{\mathrm{op}}), \mathrm{Ab})$.

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3. $\mathrm{R\text{-}Mod} \to (\mathrm{PQ}(\mathrm{R^{op}}), \mathrm{Ab})$, $M \mapsto \mathrm{Ev(M)}|_{\mathrm{PQ}(\mathrm{R^{op}})}$ is a fully faithful embedding of $\mathrm{R\text{-}Mod}$ onto $\mathrm{im}(\mathrm{PQ}(R^{\mathrm{op}}), \mathrm{Ab})$.

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Outline

Tilting objects

Definition

Let $\mathcal{G}$ be a complete, cocomplete category with a progenerator P. Then $T \in \mathcal{G}$ is an *n-tilting object* if

1. $\mathrm{pd(T)} \leq \mathrm{n}$;
2. $\mathrm{Ext^i(T^{(I)}, T^{(I)})} = 0$, for all $i > 0$, and index sets I;
3. there is a finite coresolution of P by objects in $\mathrm{Add}(T)$.

Theorem

If $M \in \mathrm{R\text{-}Mod}$ is a pure projective pure generator, then

1. $\mathrm{Ev(M)}$ is 2-tilting in $(\mathrm{mod\text{-}R}, \mathrm{Ab})$;
2. $\mathrm{Ev(M)}$ is 1-tilting in $(\mathrm{PQ}(R^{\mathrm{op}}), \mathrm{Ab})$;
3. $\mathrm{Ev(M)}$ is 0-tilting in $((\mathrm{R\text{-}mod})^{\mathrm{op}}, \mathrm{Ab})$.

Cotilting objects

Definition

Let $\mathcal{G}$ be a complete, cocomplete category with an injective cogenerator E. Then $C \in \mathcal{G}$ is an *n-cotilting object* if

1. $\mathrm{id}(C) \leq n$;
2. $\mathrm{Ext}^i(C^I, C^I) = 0$, for all $i > 0$, and index sets I;
3. there is a finite resolution of E by objects in $\mathrm{Prod}(C)$.

Theorem

If $N \in \mathrm{R\text{-}Mod}$ is a pure injective pure cogenerator, then

1. $\mathrm{Ev}(N)$ is 0-cotilting in $(\mathrm{mod\text{-}R}, \mathrm{Ab})$ (Jensen-Lenzing);
2. $\mathrm{Ev}(N)$ is 1-cotilting in $(\mathrm{PQ}(R^{\mathrm{op}}), \mathrm{Ab})$;
3. $\mathrm{Ev}(N)$ is 2-cotilting in $((\mathrm{R\text{-}mod})^{\mathrm{op}}, \mathrm{Ab})$.

The Positselski-Šťovíček correspondence

Theorem (Positselski-Šťovíček)

Let $\mathcal{G}$ be a complete, cocomplete abelian category with an injective cogenerator E. If $T \in \mathcal{G}$ is an n-tilting object in $\mathcal{G}$, then there exists a (unique) complete, cocomplete abelian category $\mathcal{G}'$ with a progenerator P' such that $\mathrm{Add}(P') \cong \mathrm{Add}(T)$ and there exists an n-cotilting object $C \in \mathcal{G}'$ such that $\mathrm{Prod(C)} \cong \mathrm{Prod(E)}$.

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