The Universal Quantum Logic of a Ring with Involution

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The Ohio State University

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Olivier's Construction

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$$A \to A^{\mathrm{ab}} := A[y_a; a \in A]/(y_a a y_a - y_a, a y_a a - a),$$

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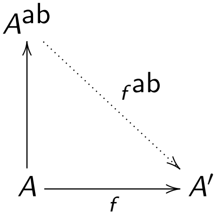

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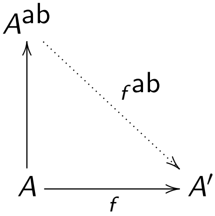

- In general, y is a **generalized inverse** of a if $aya = a$ and $yay = y$.

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- **P.M. Cohn:** Inversion height of epic R-fields and the Mal'cev kernel. (**J. Sánchez,** Localization: On Division Rings and Tilting Modules)

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  countably often: $R \to R_1^{\mathrm{MP}} \to R_2^{\mathrm{MP}} \to \cdots \to R^{\mathrm{MP}} := \varinjlim R_n^{\mathrm{MP}}$
- In general, y^* is a **Moore-Penrose inverse** of r if $ry^*r = r$, $y^*ry^* = y^*$, $ry^* = yr^*$, and $r^*y = y^*r$. An idempotent $e^2 = e$ is a **projection** if $e^* = e$.

The Lattice $\mathbb{L}(R, n)$

- A **positive primitive** formula in $\mathcal{L}(R) := (+, -, 0, r)_{r \in R}$ is one of the form

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- The lattice $\mathbb{L}(R, n) := (\mathbb{L}(R, \mathbf{u}), +, \wedge, 0, 1)$ of positive primitive fomulae in n free variables is **modular.**

The Fundamental Theorem of Projective Geometry

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- In general, $\mathbb{L}(R, n) \cong \mathbb{L}(M_n(R), 1)$, where
$$M_n(R) = \text{ the ring of } n \times n \text{ matrices over } R.$$
So if R is regular, then $M_n(R)$ coordinatizes $\mathbb{L}(R, n)$.

The Cubical Lattice $\mathbb{L}(R, \cdot)$

Definition

The sequence $\mathbb{L}(R, n)$, $n \geq 1$, is a **cubical** modular lattice: there are two pairs of morphisms

1. $\exists_i, N_i : \mathbb{L}(R, n) \to \mathbb{L}(R, n-1)$,
   $(\exists_i \varphi)(u_1, \ldots, u_{n-1}) := \exists u\ \varphi(u_1, \ldots, u_{i-1}, u, u_i, \ldots, u_{n-1})$,
   $(N_i \varphi)(u_1, \ldots, u_{n-1}) := \varphi(u_1, \ldots, u_{i-1}, 0, u_i, \ldots, u_{n-1})$;
2. $D_i^+, D_i^- : \mathbb{L}(R, n-1) \to \mathbb{L}(R, n)$,
   $(D_i^+ \psi)(u_1, \ldots, u_n) := \psi(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n)$,
   $(D_i^- \psi)(u_1, \ldots, u_n) := \psi(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n) \wedge u_i \doteq 0$.

The Prest Dual

- The **Prest dual** of a $\varphi(\mathbf{u}) = \exists \mathbf{v}\ (A, B) \begin{pmatrix} \mathbf{u}^t \\ \mathbf{v}^t \end{pmatrix} \doteq 0$ in $\mathbb{L}(R, n)$ is

$$\varphi^{\otimes}(\mathbf{u}) := \exists \mathbf{w}\ (\mathbf{u}, \mathbf{w}) \begin{pmatrix} I_n & 0 \\ A & B \end{pmatrix} \doteq 0$$

in $\mathbb{L}(R^{\mathrm{op}}, n)$. Verify that R-Mod $\models \varphi(\mathbf{u}) \leftrightarrow (\varphi^{\otimes})^{\otimes}(\mathbf{u})$.

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- **IH:** If ${}_RN \models \varphi(\mathbf{b})$ and $M_R \models \varphi^{\otimes}(\mathbf{a})$, then

$$\mathbf{a} \otimes \mathbf{b} := \sum_i a_i \otimes b_i = 0$$

  in $M \otimes N$.

The Involution

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- The map

$$\varphi(\mathbf{u}) \mapsto \varphi'(\mathbf{u}) := \exists \mathbf{w} \begin{pmatrix} I_n & A^* \\ 0 & B^* \end{pmatrix} \begin{pmatrix} \mathbf{u}^t \\ \mathbf{w}^t \end{pmatrix} \doteq 0$$

is an involution of $\mathbb{L}(R, n)$, i.e., an anti-automorphism of order 2.

Quantum logic

- A **quantum logic** is an **orthocomplemented** modular lattice $(\mathbb{L}, +, \wedge, ', 0, 1)$ with involution. It satisfies:
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- A left R-module M satisfies the Laws of Contradiction and Excluded Middle in $\mathbb{L}(R, 1)$ iff it is obtained by restriction of scalars along $R \to R^{\mathrm{MP}}$.

Definition and Examples

- **W. Wong:** Let $(k, *)$ be a field with involution and $(R, *)$ be a k-algebra with involution. A $*$-**contravariant form** on a representation ${}_RM$ of R is a $*$-sesquilinear form $B : M \times M \to k$ that satisfies
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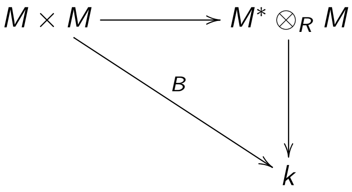

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$$\begin{array}{ccc} M \times M & \longrightarrow & M^* \otimes_R M \\ & \overset{B}{\searrow} & \downarrow \\ & & k \end{array}$$

- **The Šapovalov Form.**

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- **J.C. Jantzen:** Let $(k, *)$ be a **formally real** field with involution:

$$\sum_i x_i x_i^* = 0 \text{ implies } x_i = 0;$$

L a split **semisimple Lie algebra** over k; and $(U(L), \omega_0)$ the **universal enveloping algebra,** equipped with the $*$-linear **compact involution.** Every finite dimensional representation of L admits a definite ω_0-contravariant form.

$*$-Modules

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  1. a $*$-**module** if $\mathrm{ann}_R(M)$ is a $*$-ideal of R;
  2. a **functorial** $*$-module if $\varphi(M)/\psi(M) = 0$ iff $\psi'(M)/\varphi'(M) = 0$;
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- **IH:** Let $(k, *)$ be a field with involution and $(R, *)$ a $(k, *)$-algebra. If M is a finite dimensional simple representation of R for which $M^* \otimes_R M \neq 0$, then M is a functorial $*$-module.