Furstenberg’s intersection conjecture and the $L^q$ norm of convolutions

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**What?** Arbeitsgemeinschaft (working group): Additive Combinatorics, Entropy, and Fractal Geometry.

**When?** October 8-13, 2017

**Where?** Oberwolfach, Germany.

**Who?** Organized by E. Breuillard, M. Hochman and P.S.

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Let \( p \in \mathbb{N}_{\geq 2} \). Every point \( x \in [0, 1) \) has an expansion to base \( p \):

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x = 0.x_1x_2\ldots = \sum_{n=1}^{\infty} x_n p^{-n}, \quad x_i \in \{0, 1, \ldots, p-1\}.
\]

**Basic facts:**

1. All but countably many (rational) points have a unique expansion; the remaining ones have two expansions.

2. A point is rational if and only if the expansion is eventually periodic.

3. Expansions in bases \( p^n \) and \( p^k \) are “almost the same” (look at base \( p \) in blocks of length \( n \) and \( k \)).
Base $p$ expansions

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Multiplication by $p$

Definition

For $p \in \mathbb{N}_{\geq 2}$, let

$$T_p = px \mod 1$$

be multiplication by $p$ on the circle.

Symbolically, $T_p x$ corresponds to shifting the $p$-ary expansion $x$: there is a factor map, which is one-to-one outside of the countably many points with two $p$-ary expansions.
Multiplying by 2 and by 3: the founding father
Some of the areas that Furstenberg initiated

1. Ergodic theoretic methods in combinatorics (ergodic proof of Szemerédi’s Theorem,...).
2. Products of random matrices, non-commutative ergodic theory (simplicity of Lyapunov exponents, ...).
3. Unique ergodicity of horocycle flow, toral maps, ...
4. Disjointness of dynamical systems.
5. $\times 2$, $\times 3$, rigidity of higher order actions.
6. Fractal geometry $\cap$ ergodic theory (CP-processes, ...).
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Expansions in different bases

Principle (Furstenberg)

Expansions in bases 2 and 3 have no common structure. More generally, this holds for bases $p$ and $q$ which are not powers of a common integer or, equivalently, $\log p / \log q$ is irrational.

Remark

Furstenberg proved some results, and proposed many conjectures, which make precise (in different ways) the concept of “no common structure”.

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Invariant sets

Definition

A set $A \subset [0, 1)$ is $T_p$-invariant if $T_p(A) \subset A$. That is, shifting the $p$-ary expansion of a point in $A$ gives another point in $A$.

- If $p$ and $q$ are coprime, then $\{1/q, \ldots, (q - 1)/q\}$ is $T_p$-invariant.
- $[0, 1)$ is $T_p$-invariant.
- Let $D \subset \{0, 1, \ldots, p - 1\}$. The set $A = A_{p,D}$ of points whose base $p$-expansion has only digits from $D$ is $T_p$-invariant. We call it a $p$-Cantor set. Example: the middle-thirds Cantor set.
- There is a wild abundance of invariant sets and no classification or description is possible.
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Invariant sets and shared structure

Principle (Furstenberg, slightly more concrete version)

If $A, B$ are closed and invariant under $T_2, T_3$ respectively, then $A$ and $B$ have no common structure.

Theorem (Furstenberg (1967))

If $A$ is jointly invariant under $T_2$ and $T_3$, then $A$ is either finite or the whole circle $[0, 1)$.

Remarks

- The theorem is a weak confirmation of the principle since the set $A$ and itself certainly have a lot of common structure!
- One should think of finite sets and the whole circle as sets "without structure".

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A corollary in terms of orbits

Observation

- If $x$ is rational, then the orbit $\{T_2^n T_3^m x\}^{\infty}_{n,m=1}$ is finite.
- If $x$ is irrational, then the orbit $\{T_2^n T_3^m x\}^{\infty}_{n,m=1}$ is infinite (and its closure is invariant under $T_2$ and $T_3$).

Corollary (Furstenberg 1967)

If $x$ is irrational, then the orbit $\{T_2^n T_3^m x\}^{\infty}_{n,m=1}$ is dense in $[0, 1)$.
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Definition

A Borel probability measure $\mu$ on $[0, 1)$ is $T_p$-invariant if

$$\mu(B) = \mu(T_p^{-1}B)$$

for all Borel sets $B$.

Conjecture (Furstenberg 1967)

If $\mu$ is $T_2$ and $T_3$ invariant, then $\mu$ is a convex combination of Lebesgue measure and an atomic measure supported on rationals.
“The” $\times 2, \times 3$ Furstenberg conjecture

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How to quantify “shared structure”

1 Furstenberg’s Theorem says that non-trivial $T_2$ and $T_3$ invariant sets do not have too much shared structure in the most basic sense: they cannot be equal.

2 How can we quantify shared structure in finer/more quantitative ways? The sets we are interested in are fractal: they are uncountable but of zero Lebesgue measure, and have some form of (sub)-self-similarity.

3 Geometry helps quantify common structure. If two sets $A, B \subset \mathbb{R}$ have no shared structure then the intersection $A \cap B$ should be “as small as possible” (perhaps even after distorting $A$ and/or $B$ in some way).
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Hausdorff Dimension

- Best exponent for coverings of the set by balls of arbitrary (possibly different) radii:
  \[ \dim_H(A) = \inf \left\{ s : \inf \sum_i r_i^s : A \subset \bigcup_i B(x_i, r_i) \right\} = 0 \]

- Gives a notion of “size” for sets in \( \mathbb{R}^d \), varies between 0 and \( d \), gives the right size to smooth objects, is invariant under bi-Lipschitz maps, is countably stable, assigns size \( \log 2 / \log 3 \) to the middle-thirds Cantor set,...

- If \( A \subset T \) is \( T_p \)-invariant, then \( \dim_H A = \frac{h_{\text{top}}(A)}{\log p} \).

- If \( A = A_{p,D} \) is a \( p \)-Cantor set, then \( \dim_H A = \frac{\log |D|}{\log p} \).
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Dimensions of intersections

**Question**

If $A, B \subset \mathbb{R}^d$, what do we expect $\dim_H(A \cap B)$ to be “typically”?

**Remark**

If $A, B$ are affine planes in $\mathbb{R}^d$ in general position, then

$$\dim(A \cap B) = \min(\dim(A) + \dim(B) - d, 0).$$

**Theorem (Marstrand 1954)**

If $A, B \subset \mathbb{R}$ are “nice” sets, then for almost all affine maps $f : \mathbb{R} \to \mathbb{R}$,

$$\dim_H(A \cap f(B)) \leq \min(\dim_H(A) + \dim_H(B) - 1, 0),$$

and this does not hold for any smaller number on the RHS.
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and this does not hold for any smaller number on the RHS.
Conjecture (Furstenberg 1969)

Let $A$, $B$ be closed and invariant under $T_p$, $T_q$ (seen as subsets of $\mathbb{R}$). Then for every affine bijection $f : \mathbb{R} \to \mathbb{R}$,

$$\dim_H(A \cap f(B)) \leq \max(\dim_H(A) + \dim_H(B) - 1, 0).$$
Remarks

Furstenberg’s intersection conjecture gave rise to the study of “Furstenberg sets”, containing a set of dimension \( \alpha \) in (almost-)every direction. Finding the smallest possible dimension is such sets is a wide open problem.

Theorem (Furstenberg 1969, Wolff 2000)

The conjecture holds if \( \dim_H(A) + \dim_H(B) \leq 1/2 \). More generally, one always has

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No example of invariant sets \( A, B \) for which the conjecture holds with \( \dim_H(A) + \dim_H(B) > 1/2 \) was known.
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Solution to Furstenberg’s intersection conjecture

Theorem (P.S. 2016)

Furstenberg’s intersection conjecture is true.

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Meng Wu independently found another proof. The proofs are completely different. Wu’s proof is purely ergodic theoretical, using CP-processes (introduced by Furstenberg in the paper where he stated the conjecture) and Sinai’s factor theorem.

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A picture!

\[ A \times B. \]
$A \times B \cap \text{diagonal} = A \cap B.$
$A \times B \cap \text{any line} = A \cap \text{affine image of } B.$
Tools involved in the proof

1. **Additive combinatorics**: an inverse theorem for the $L^q$ norm of the convolution of two finitely supported measures (Balog-Szemerédi-Gowers Theorem, Bourgain’s additive part of discretized sum-product results).

2. **Ergodic theory**: key role played by subadditive cocycle over an irrational rotation (cocycle borrowed from Nazarov-Peres-S. 2012, uses the proof of the subadditive ergodic theorem given by Katznelson-Weiss).

3. **Multifractal analysis** ($L^q$ spectrum, regularity at points of differentiability).

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From now on a measure is a probability measure supported on $2^{-m} \mathbb{Z} \cap [0, 1) = \{ j2^{-m} : 0 \leq j < 2^m \}$ for some large $m$.

The $L^q$ norm of $\mu$ ($q \geq 1$) is

$$\| \mu \|_q^q = \sum_x \mu(x)^q, \quad \| \mu \|_\infty = \max_x \mu(x).$$

$$2^{-m/q'} \leq \| \mu \|_q \leq 1,$$

with a “small” $L^q$ norm corresponding to “uniform” measures and a large $L^q$ norm to “localized” measures.
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$L^q$ norms of convolutions

The convolution of $\mu, \nu$ is

$$(\mu \ast \nu)(x) = \sum_{a+b=x} \mu(a)\nu(b).$$

(Addition modulo 1, although it makes no difference)

Young’s inequality (just convexity of $t \mapsto t^q$)

$$\|\mu \ast \nu\|_q \leq \|\mu\|_q \|\nu\|_1 = \|\mu\|_q.$$ 

When is there (almost) equality in Young’s inequality? (for $1 < q < \infty$). Two easy situations:

1. $\mu$ is (almost) uniform.
2. $\nu$ is (almost) an atom.

There are less trivial examples: let $A$ be a set that is “uniform” on some scales and “an atom” at the complementary scales. Then $\mu = 1_A/|A|$ satisfies $\|\mu \ast \mu\|_q \sim \|\mu\|_q$. 

P. Shmerkin (U.T. Di Tella/CONICET)
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An inverse theorem for the flattening of $L^q$ norms

Theorem (Informal version)

Let $\mu, \nu$ be measures such that

$$\|\mu \ast \nu\|_q \geq 2^{-\varepsilon m} \|\mu\|_q.$$ 

Then there are “regular” sets $A, B$ of “large” $\mu, \nu$-measure such that in a “multiscale decomposition”, on each scale either “$A$ is almost uniform” or “$B$ is an atom”.

P. Shmerkin (U.T. Di Tella/CONICET)
Trees, branching, regular sets

**Definition**

Suppose $m = \ell m'$ for some (large) $\ell$, $m'$. Given a set $A \subset m\mathbb{Z} \cap [0,1)$, we consider the associated base-$2^\ell$ tree $T_A$: its vertices of level $j$ are those dyadic intervals $I$ of length $(2^{-\ell})^j$ that intersect $A$.

**Definition**

Given a sequence $k = (k_1, \ldots, k_{m'})$ with $k_i \in \{1, \ldots, \ell\}$, we say that $A$ is $k$-regular if the following holds:

For each dyadic interval of $I$ of length $2^{-j\ell}$ that intersects $A$, there are exactly $k_{j+1}$ intervals $J$ of length $2^{-(j+1)\ell}$ that intersect $A \cap I$.

In other words, for the tree $T_A$, each vertex of level $j$ has exactly $k_{j+1}$ children.
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The inverse theorem with more details

Theorem (P.S. 2016)

Given $\delta > 0$, there is $\varepsilon > 0$ such that the following holds for $\ell, m'$ large enough. Let $m = \ell m'$. If

$$\|\mu * \nu\|_q \geq 2^{-\varepsilon m} \|\mu\|_q,$$

then there are sets $A, B$ such that:

- $\|\mu|_A\|_q \geq 2^{-\delta m} \|\mu\|_q, \nu(B) \geq 2^{-\delta m} \|\nu\|_1$.
- $\mu(x) \leq 2\mu(y)$ for all $x, y \in A$, same for $\nu$ and $B$.
- $A$ and $B$ are $k$-regular and $k'$ regular respectively for some sequences $(k_1, \ldots, k_{m'})$, $(k'_1, \ldots, k'_{m'})$.
- For each $j$, Either $k_j \geq 2^{(1-\delta)\ell}$ or $k'_j = 1$. 

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A corollary

Definition

A set $B \subset [0, 1]$ is $\eta$-porous if for every interval $I \subset [0, 1]$ there is an interval $J \subset I \cap [0, 1] \setminus B$ with $|J| \geq \eta|I|$.

If $B \subset 2^{-m}\mathbb{Z} \cap [0, 1]$, then we only require this for $|I| \geq 2^{-m}/\eta$.

Corollary

If $\text{supp}(\mu)$ is $\eta$-porous, then either

$$\|\nu\|_q \geq 2^{-\delta m},$$

or

$$\|\mu \ast \nu\|_q \leq 2^{-\varepsilon m}\|\mu\|_q,$$

where $\varepsilon = \varepsilon(\eta, \delta, q) > 0$.

In particular, this holds if $\mu$ is a (discretization of) an Ahlfors-regular measure, generalizing a result of Dyatlov-Zahl.
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Two main tools in the proof of the inverse theorem

Asymmetric Balog-Szemerédi-Gowers Theorem (Tao-Vu): If \(A, B \subset 2^{-m}\mathbb{Z} \cap [0, 1)\) are such that

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then there are subsets \(A' \subset A, B' \subset B\) such that \(|A'| \geq 2^{-\varepsilon}|A|, |B'| \geq 2^{-\varepsilon}|B|,\) and

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Bourgain’s additive part of sum-product machinery: If \(|A' + A'| \leq 2^{\varepsilon m}|A'|\), then \(A'\) contains a \(k\)-regular subset \(A''\) such that:

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Many thanks!!!