**Problem 1:**

(1). Notice this is a rectangular (prism) region, so the integral is

$$\int_0^{\ln 2}\int_1^3\int_0^2 yze^x dzdydx = \left(\int_0^2 zdz\right)\left(\int_1^3 ydy\right)\left(\int_0^{\ln 2} e^x dx\right),$$

and since $e^0 = 1$ and $e^{\ln 2} = 2$, this is

$$\left(\frac{1}{2}z^2\Big|_0^2\right)\left(\frac{1}{2}y^2\Big|_1^3\right)\left(e^x\Big|_0^{\ln 2}\right) = 2\cdot 4\cdot 1 = 8.$$

(2). As written in the integral, we have $0 \le y \le \pi$ and $y \le x \le \pi$, meaning our region is bounded by $y = x$ on the left and $x = \pi$ on the right, and we are sweeping up from $y = 0$ to $y = \pi$. So, our region is the triangle with vertices at $(0, 0)$, $(\pi, 0)$, and $(\pi, \pi)$. If we want to integrate with respect to $x$ first, we then notice looking at this triangle that our region is bounded above by the line $y = x$ and below by the line $y = 0$, and we are considering values of $x$ between 0 and $\pi$. Thus, the integral is

$$\int_0^{\pi}\int_0^{x} 2\cos(x^2)dydx.$$

One note about Problem 1 (1): It was a very common error to have $\int_0^{\ln 2} e^x = e^{\ln 2}$. You must ALWAYS be careful with 0 as a bound of integration - you can't just assume that gives you 0 as the function value when you use the Fundamental Theorem of Calculus.

Two notes about Problem 1 (2): First, the integrand has NOTHING to do with the region of integration. A few of you gave me sketches of trig function graphs, and you shouldn't have because no trig functions occur in your bounds of integration. Second, a number of you gave me the triangle with vertices at $(0, 0)$, $(0, \pi)$, and $(\pi, \pi)$. This is erroneous because we have $y \le x \le \pi$, not $0 \le x \le y$.

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**Problem 2:** We let $f(x, y) = 1 + \frac{1}{2}x^2 + 3y^2$.

(a). $\nabla f(x, y) = \langle f_x, f_y\rangle = \langle x, 6y\rangle$, so $\nabla f(2, 1) = \langle 2, 6\rangle$.

(b). We need unit vectors to compute directional derivatives. $\langle -4, 3\rangle$ is not a unit vector, but it has length 5, so we use the unit vector $\mathbf{u} = \langle -4/5, 3/5\rangle$. The directional derivative of $f$ at (2,1) in the direction of $\langle -4, 3\rangle$ is therefore $D_{\mathbf{u}}f(2, 1) = \mathbf{u}\cdot\nabla f(2, 1) = \langle -4/5, 3/5\rangle \cdot \langle 2, 6\rangle = 2$.

(c). The unit vector $\mathbf{v}$ that gives the direction of steepest ascent at $(2, 1)$ is the unit vector in the direction of $\nabla f(2, 1)$. Since $|\langle 2, 6\rangle| = \sqrt{40}$, $\mathbf{v} = \frac{1}{\sqrt{40}}\langle 2, 6\rangle$.

(d). For the surface $F(x, y, z) = f(x, y) - z = 1 + \frac{1}{2}x^2 + 3y^3 - z = 0$, a normal vector $\mathbf{n}$ at the point $P(2, 1, 6)$ is $\nabla F(2, 1, 6) = \langle 2, 6, -1\rangle$.

**Problem 3:**

(a). $f_x = 2x - 1$ and $f_y = -8y$, so $f_{xx} = 2$, $f_{xy} = 0$, and $f_{yy} = -8$. Thus, $D(1,0) = f_{xx}(1,0)f_{yy}(1,0) - f_{xy}(1,0)^2 = 2(-8) - 0^2 = -16 < 0$, so $P(1,0)$ is a saddle point.

(b). $u = \frac{x}{y}$ and $v = y$, so $y = v$, and $x = yu = uv$. Thus, $J(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$.

(c). The line segment from $(0,0)$ to $(1,1)$ is parametrized by $\mathbf{r}(t) = \langle t, t \rangle$, $0 \leq t \leq 1$, so $\mathbf{r}'(t) = \langle 1, 1 \rangle$ and $|\mathbf{r}'(t)| = \sqrt{2}$. This means that since $ds = |\mathbf{r}'(t)|dt = \sqrt{2}dt$, $\int_C (x+y)ds = \int_0^1 (t+t)\sqrt{2}dt = \sqrt{2}\int_0^1 (2t)dt = \sqrt{2}$.

(d). Since $\mathbf{F}$ has the potential function $\varphi$, $\mathbf{F}$ is conservative, meaning the fundamental theorem of calculus for line integrals applies. $C$ starts at $\mathbf{r}(0) = (0, 2 \cdot 0) = (0,0)$ and ends at $\mathbf{r}(1) = (1, 2 \cdot 1) = (1,2)$. Therefore, the fundamental theorem of calculus for line integrals says $\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(1,2) = \varphi(0,0) = (1 \cdot 2) - (0 \cdot 0) = 2$.

(e). Converting to spherical coordinates, $D$ is given by

$$\{(\rho, \varphi, \theta) : \ 0 \leq \rho \leq 1, 0 \leq \varphi \leq \pi, 0 \leq \pi \leq 2\pi\},$$

$\sqrt{x^2 + y^2 + z^2} = \rho$, and $dV = \rho^2 \sin\varphi d\rho d\varphi d\theta$. Therefore,

$$\iiint_D \sqrt{x^2 + y^2 + z^2}\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^3 \sin\varphi d\rho d\varphi d\theta.$$

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**Problem 4:** Our constraint is $g(x,y) = x^2 - 3xy + 4y^2 = 14$, so $\nabla f = \lambda \nabla g$ becomes $\langle 1, -2 \rangle = \lambda \langle 2x - 3y, 8y - 3x \rangle$. This tells us $1 = \lambda(2x - 3y)$ and $-2 = \lambda(8y - 3x)$. Since we cannot have $2x - 3y = 0$ and $8y - 3x$ in these equations (otherwise $1 = 0$ and $-2 = 0$!), we can divide by them to isolate $\lambda$ on both sides, getting us $\frac{1}{2x-3y} = \lambda = \frac{-2}{8y-3x}$. Cross multiplying, $-2(2x - 3y) = 8y - 3x$ and therefore $x = -2y$. Substituting this into the constraint $g(x,y) = 14$, we get $4y^2 + 6y^2 1 + 4y^2 = 14$, meaning $14y^2 = 14$ and $y = \pm 1$. If $y = 1$, $x = -2$, and if $y = -1$, $x = 2$. $f(2,-1) = 4$ and $f(-2,1) = -4$, so the method of Lagrange multipliers tells us that the maximum and minimum values of $f$ on the constraint curve $g(x,y) = 14$ are respectively 4 and $-4$.

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**Problem 5:**

We want to find the volume of the solid bounded between the paraboloid $z = 4 - x^2 - y^2$ and the $xy$-plane. From the picture, our top $z$ is $4 - x^2 - y^2$, and our bottom $z$ is the $xy$-plane, $z = 0$. Also, from the picture, it is also apparent that the solid is widest at the bottom, and therefore the boundary of the projection of the solid onto the $xy$-plane is therefore $4 - x^2 - y^2 = 0$, i.e. $x^2 + y^2 = 4$, the circle of radius 2 centered at the origin. Therefore, the projection of the solid onto the $xy$-plane is $R = \{(x,y) : \ x^2 + y^2 \leq 4\} = \{(r,\theta) : \ 0 \leq r \leq 2, 0 \leq r \leq 2\pi\}$. This means the volume of our solid is $\iint_R (4 - x^2 - y^2) - (0)dA = \int_0^{2\pi} \int_0^2 (4 - r^2)(r dr d\theta) = 2\pi \left(2r^2 - \frac{1}{4}r^4\right)\Big|_0^2 = 2\pi(8 - 4) = 8\pi$.

**Problem 6:**

(a). True. For $\mathbf{F} = \langle f, g \rangle = \langle ye^x + x^2, e^x + \sin y \rangle$, $f_y = e^x$ and $g_x = e^x$, and therefore $f_y = g_x$, meaning $\mathbf{F}$ is conservative on $\mathbb{R}^2$.

(b). True. Since $C$ is the unit circle $x^2 + y^2 = 1$, it's a level curve for $g(x, y) = 1$, meaning $\nabla g = \langle 2x, 2y \rangle$ is normal to $C$ everywhere. Therefore, $\mathbf{F} = \langle x, y \rangle = \frac{1}{2}\nabla g$ is normal to $C$ everywhere, meaning the tangential component of $\mathbf{F}$ on $C$ is zero everywhere. Therefore $\mathbf{F} \cdot T = 0$ everywhere on $C$, so the circulation $\int_C \mathbf{F} \cdot \mathbf{T} ds$ is zero.

(c). False. Flux integrals involve $\mathbf{n}$ in the dot product with $\mathbf{F}$, so the integrand should be $\mathbf{F} \cdot \mathbf{n}$, not $\mathbf{F} \cdot \mathbf{n}$.

(d). True. If $R$ is given by $x$ between 2 functions of $y$ and $y$ between 2 numbers, then we integrate $x$ first, keeping these range of "values" given as our bounds of integration for the area of $R$. Since we're integrating $dxdy$, $dA = dxdy$, and our integrand is 1.