13.8, Problem 45: Find the absolute maximum and minimum values of $f(x,y) = 4+2x^2+y^2$ on the region $R = \{(x,y) : -1 \le x \le 1,\ -1 \le y \le 1\}$.

Solution: Notice that f as a polynomial is continuous and R is a closed and bounded region. Therefore, our modified Extreme Value Theorem applies and tells us that absolute extrema for f exist on R, and they occur either in the interior at critical points or on the boundary. $f_x = 4x$ and $f_y = 2y$, so our only critical point is $(0,0)$, which is in the interior. $f_{xx} = 4$, $f_{xy} = 0$, and $f_{yy} = 0$, so $D(0,0) = 8 > 0$, and $f_{xx}(0,0) = 4 > 0$, so $(0,0)$ is thus a local minimum. Now we test the boundary: the boundary is a square, so we consider each segment separately. On the left side, $x = -1$ and $-1 \le y \le 1$, so $f(x,y) = 6 + y^2$, maximmizing at $y = \pm 1$ and minimizing at $y = 0$, where $f(-1,0) = 6$. The other sides are similar: f maximizes at the corners and minimizes on the boundary at $(0,1)$ and $(0,-1)$. However, f is 5 here and 4 at $(0,0)$. Hence, f has an absolute minimum of 4 at $(0,0)$ on R.

13.8, Problem 53: If possible, find the absolute maximum and/or minimum values of the function $f(x,y) = x^2 + y^2 - 4$ on the region $R = \{(x,y) :\ x^2 + y^2 < 4\}$.

Solution: Geometrically, R is the interior of the circle of radius 2 centered at $(0,0)$. f has a critical point at $(0,0)$, and $f(0,0) = -4$. Since $x^2 + y^2 > 0$ when either x or y is nonzero, we then konw $f(x,y) > -4$ elsewhere on R, so f has an absolute minimum of -4 at $(0,0)$ on R. Now we ask: what about the absolute maximum? Observe that increasing $x^2 + y^2$ (i.e. moving outward from the origin) increases f, and we can bring $x^2 + y^2$ as close to 4 as we want but can't have it actually reach 4 itself because we're dealing with an open disk. So, f would maximize at 0 on the boundary, but since the boundary is completely omitted, no absolute maximum on R exists for f.

13.8, Problem 58: Find the point(s) on the cone $z^2 = x^2 + y^2$ nearest the point $P(1,4,0)$.

Solution: We want to minimize distance, or equivalently, distance squared from $(1,4,0)$. Let d^2 denote distance squared. Then, $d^2(x,y,z) = (x-1)^2+(y-4)^2+z^2$, and since we're only considering points on the cone $z^2 = x^2+y^2$, this is $d^2(x,y) = (x-1)^2 + (y-4)^2 + x^2 + y^2$. Now we can apply our second derivative test, starting by looking for critical points. $d^2_x = 2(x-1) + 2x = 4x - 2$ and $d^2_y = 2(y-4) + 2y$, so if $d^2_x = d^2_y = 0$, then $x = 1/2$ and $y = 2$. $d^2_{xx} = 4$, $d^2_{xy} = 0$, and $d^2_{yy} = 4$, so by the second derivative test, $(1/2, 2, \pm\sqrt{17}/2)$ are local minima for d^2 and hence d. With an unbounded surface like a cone, no furthest point (i.e. absolute maximum for d^2) can exist, and obviously, a local minimum must be an absolute minimum, since that's how distance works: given an surface and a point, there must be points on the surface closest to that point and moving away from the local minima, the distance to $(1,4,0)$ increases without bound.