

1 Lagrange Multipliers

In this section, we seek the maximum and minimum values of an **objective function** $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ (or $f : \mathbb{R}^3 \rightarrow \mathbb{R}$) with the restriction that our variables x and y (and z if f has 3 variables) must lie on a **constraint curve** (resp., surface) C in the xy -plane (resp., 3D space) given by $g(x, y) = 0$ (resp., $g(x, y, z) = 0$).

Theorem 1.1 *Let f be a differentiable function in a region of \mathbb{R}^2 containing a smooth curve C given by $g(x, y) = 0$. Assume that f has a local extreme value on C at a point $P(a, b)$. Then, $\nabla f(a, b)$ is orthogonal to the line tangent to C at P . It then follows that if $\nabla g(a, b) \neq 0$ there is a real number λ (called a **Lagrange multiplier**) such that $\nabla f(a, b) = \nabla g(a, b)$.*

The result for 3 variables is completely analogous. The second part of the result follows from the fact that **gradients are always orthogonal to level curves**. Thus, we have a theorem:

Theorem 1.2 *If a point (a, b) is a max or min for f on a curve C given by $g(x, y) = 0$, then a Lagrange multiplier equation $\nabla f(a, b) = \lambda \nabla g(a, b)$ is satisfied for some $\lambda \neq 0$.*

We consider when such an equation is satisfied:

Procedure 1.3 *Let the objective function f and the constraint function g be differentiable on a region of \mathbb{R}^2 with $\nabla g \neq 0$ on the curve $g(x, y) = 0$. To locate the extreme values of f subject to the constraint $g(x, y) = 0$, carry out the following steps:*

1. Find the values of x, y , and λ (if they exist) that satisfy BOTH $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$.
2. Among the values (x, y) found in Step 1, select the pairs (x, y) corresponding to the largest and smallest function values. These are the maximum and minimum values of f subject to $g(x, y) = 0$.

Admittedly, **1.** doesn't tell you how to go about finding x, y , and λ , and **that's the hardest part of Lagrange multiplier problems**. So, here are some tips.

Tip 1.4 *From $\nabla f(x, y) = \lambda \nabla g(x, y)$, you get 2 equations $f_x(x, y) = \lambda g_x(x, y)$ and $f_y = \lambda g_y(x, y)$. To find the values of x, y , and λ satisfying them (actually, **you don't really care about solving for λ** if you can get the corresponding x and y), you generally want to do one of the following:*

- (a). If $f_x = \lambda g_x$ is an equation of one variable and $f_y = \lambda g_y$ is an equation of one variable, then you may find it helpful to use the **zero product property** from high school algebra (if $ab = 0$, then $a = 0$ or $b = 0$) to break things up into cases (an example of this approach is given in Example 2 on page 988 of the textbook). **DO NOT** forget to consider the constraint equation.
- (b). Depending on the setup, you may choose to eliminate λ . This may be done by either solving for λ in both equations and then setting the two λ values equal, **OR** you may multiply both equations by some monomial to get the least common multiple of λg_x and λg_y on both sides of the equations, allowing you to set the remaining parts of the equations equal to each other (this is best applied in an example like 1.9 #23 or #28, for instance).

WARNING 1.5 *If the given point(s) satisfies the Lagrange Multipliers equation $\nabla f = \lambda \nabla g$, then you only know that the point is a max or min of f on the curve or surface given by the constraint $g(x, y) = 0$ (or $g(x, y, z) = 0$). You will need to justify why it's a max and not a min or why it's a min and not a max.*

WARNING 1.6 *When looking for extrema on a closed and bounded region R , the Lagrange Multiplier procedure given above is **VERY** useful when used in conjunction with the Strengthened Extreme Value Theorem. However, the points found with Lagrange Multipliers are maxima/minima **ONLY** on the boundary. They **MUST** be compared with the maxima/minima found in the interior (which must be critical points) before you can conclude the points found with Lagrange Multipliers are extrema for the **WHOLE** region.*

2 Double Integrals over Rectangular Regions

Procedure 2.1 The double integral $\iint_R f(x, y) dA$ over a rectangular region $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$ may be thought of as a “net volume” between the surface $z = f(x, y)$ and the region R in the xy -plane, where $z > 0$ contribute positive volume and $z < 0$ contribute negative volume. This can be conceptualized by the **general slicing method** (see section 6.3 in the text): along the segment $x = x_0$ $a \leq x_0 \leq b$ in R , $z = f(x, y)$ and the xy -plane form a 2D shape of area $A(x) = \int_c^d f(x, y) dy$, so the volume of the total region by the general slicing method is therefore

$$V = \int_a^b A(x) dx = \int_a^b \int_c^d f(x, y) dy dx.$$

Matching geometric intuition, the order can be reversed provided f is continuous.

Theorem 2.2 [Fubini] Let f be continuous on the rectangular region $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. The double integral of f over R may be evaluated by either of the two iterated integrals:

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx.$$

Tip 2.3 The following are helpful tips and shortcuts for double integral computation:

(a). In light of Fubini’s Theorem, it’s worth noting that sometimes one order of integration can sometimes be MUCH easier to compute than the other. **ALWAYS** try to see if the integrand is a partial derivative of another function. If f is the partial derivative of some function g with respect to y , you should integrate with respect to y first, as the fundamental theorem of calculus from first semester calculus will apply.

(b). Let R be as above. If $f(x, y)$ may be written as a product $g(x)h(y)$, where g is a function of just x and h is a function of just y , then we may rewrite the double integral as a product of single integrals

$$\iint_R f(x, y) dA = \left(\int_a^b g(x) dx \right) \left(\int_c^d h(y) dy \right).$$

3 Average Value of a Function in a Region

Recall from Section 5.4 that for an integrable $f : \mathbb{R} \rightarrow \mathbb{R}$, the average value of f on the interval $[a, b]$ is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Analogously, for an integrable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, the average value of f on R is

$$\bar{f} = \frac{1}{\text{area of } R} \iint_R f(x, y) dA.$$

For a general region R in the xy -plane,

$$\text{area of } R = \iint_R dA.$$