Math 2153 - Spring 2017

Name: ______________________

Quiz 9 - Take Home (10 pts)

Recitation Time: ______________________

SHOW ALL WORK!!! Unsupported answers might not receive full credit. Furthermore, please give me EXACT answers.

Problem 1 [4 pts] Let D be the solid region bounded between the upper half of the ellipsoid $\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} = 1$ and the plane $z = 0$.

(a). [0.5 pts] Find a transformation $T_1 : S \to D$ with $(u, v, w) \mapsto (x, y, z)$ where S is the upper half of the unit ball centered at the origin.

$$x = 3u$$
$$y = 4v$$
$$z = 2w$$

(b). [0.5 pts] As we learned in Section 14.5, S may be expressed in spherical coordinates via $u = \rho\sin\varphi\cos\theta$, $v = \rho\sin\varphi\sin\theta$, and $w = \rho\cos\varphi$. This describes a transformation $T_2 : R \to S$ from a region R in (ρ, φ, θ) 3D-space into S. Give a description of R in (ρ, φ, θ) 3D-space.

$$0 \le \rho \le 1$$
$$0 \le \varphi \le \frac{\pi}{2}$$
$$0 \le \theta \le 2\pi$$

(c). [0.5 pts] Use the previous parts to express explicitly the transformation $T_1 \circ T_2 : R \to D$, $(\rho, \varphi, \theta) \mapsto (x, y, z)$.

$$x = 3\rho\sin\varphi\cos\theta$$
$$y = 4\rho\sin\varphi\sin\theta$$
$$z = 2\rho\cos\varphi$$

(d). [1 pt] Find the Jacobian for the transformation $T_1 \circ T_2$.

$$J = \begin{vmatrix} x_\rho & x_\varphi & x_\theta \\ y_\rho & y_\varphi & y_\theta \\ z_\rho & z_\varphi & z_\theta \end{vmatrix} = \begin{vmatrix} 3\sin\varphi\cos\theta & 3\rho\cos\varphi\cos\theta & -3\rho\sin\varphi\sin\theta \\ 4\sin\varphi\sin\theta & 4\rho\cos\varphi\sin\theta & 4\rho\sin\varphi\cos\theta \\ 2\cos\varphi & -2\rho\sin\varphi & 0 \end{vmatrix}$$

$$= 24\rho^2\sin^3\varphi\cos^2\theta + 24\rho^2\sin\varphi\cos^2\varphi\cos^2\theta - 3\rho\sin\varphi\sin\theta(-8\rho\sin^2\varphi\sin\theta - 8\rho\cos^2\varphi\sin\theta)$$

$$= 24\rho^2\sin^3\varphi\cos^2\theta + 24\rho^2\sin\varphi\cos^2\varphi\cos^2\theta + 24\rho^2\sin\varphi\sin^2\theta \quad 1$$

$$= 24\rho^2\sin\varphi(\sin^2\varphi\cos^2\theta + \cos^2\varphi\cos^2\theta + \sin^2\theta)$$

(e.) [1.5 pts] Find $\iiint_D z\,dV$.

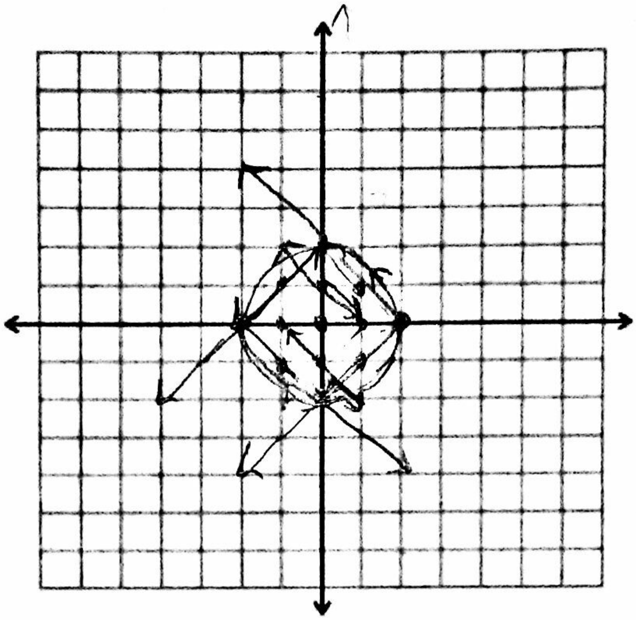
$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 (2\rho\cos\varphi)(24\rho^2\sin\varphi)\,d\rho\,d\varphi\,d\theta$$

$$96\pi\left(\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi\,d\varphi\right)\left(\int_0^1\rho^3\,d\rho\right) = (96\pi)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right) = \boxed{12\pi}$$

$$\left(\tfrac{1}{2}\sin^2\varphi\right)\Big|_0^{\frac{\pi}{2}}$$

Problem 2 [5 pts] Consider the vector field $\mathbf{F} = \langle y - 2x, 2x - y\rangle$ and the curve C given by $\mathbf{r}(t) = \langle 2\cos t, 2\sin t\rangle$, $0 \le t \le 2\pi$.

(a). [2 pts] Sketch C (indicating orientation) and 12 vectors in the vector field for $\mathbf{F}$. Include at least 4 vectors with tails at points on C.



$(0,2) \mapsto \langle 2,-2\rangle$
$(2,0) \mapsto \langle -4,4\rangle$
$(-2,0) \mapsto \langle 4,-4\rangle$
$(0,-2) \mapsto \langle -2,2\rangle$
$(1,1) \mapsto \langle -1,1\rangle$
$(1,-1) \mapsto \langle -3,3\rangle$
$(-1,1) \mapsto \langle 3,-3\rangle$
$(-1,-1) \mapsto \langle 1,-1\rangle$
$(0,1) \mapsto \langle 1,-1\rangle$
$(1,0) \mapsto \langle -2,2\rangle$
$(-1,0) \mapsto \langle 2,-2\rangle$
$(0,-1) \mapsto \langle -1,1\rangle$

(b). [1 pt] Make a prediction as to whether the flux will be positive negative, or zero, and explain your prediction using the intuitive meaning of flux. for the 4 pts on C

Flux appears to be negative since the vectors seem to point inside the curve, and that appears to be the general pattern.

(c). [2 pts] Compute the flux of $\mathbf{F}$ on C.

$\vec{r}'(t) = \langle -2\sin t, 2\cos t\rangle$.

$$\text{flux} = \int_C \vec{F}\cdot\vec{n}\,ds = \int_0^{2\pi} (y(t) - 2x(t))\,y'(t) - (2x(t) - y(t))\,x'(t)\,dt$$

$$= \int_0^{2\pi} (2\sin t - 4\cos t)(2\cos t) - (4\cos t - 2\sin t)(-2\sin t)\,dt$$

$$= \int_0^{2\pi} 4\sin t\cos t - 8\cos^2 t + 8\sin t\cos t - 4\sin^2 t\,dt$$

$$= \int_0^{2\pi} (6\sin(2t) - 4 - 4\cos^2 t)\,dt = \int_0^{2\pi} 6\sin 2t - 6 - 2\cos(2t)\,dt \qquad (-3-12\pi)-(-3)$$

$$\cos^2 t = \frac{1+\cos(2t)}{2} \qquad = \Big[-3\cos 2t - 6t - \sin 2t\Big]_0^{2\pi} = \boxed{-12\pi}$$

Problem 3 [1 pt] Suppose that $g : \mathbb{R}^2 \to \mathbb{R}$ is the potential function for a vector field $\mathbf{F}$.

(a). [0.5pts] State the simple but important fact about the vectors $\mathbf{F}(x, y)$ evaluated at points (x, y) on the level curves for g.

$\vec{F}(x,y)$ will be perp to the tangent vector

(b). [0.5pts] Consider one level curve C given by $g(x, y) = D$. Based on the intuitive definition of circulation, what do you think the circulation of $\mathbf{F}$ on C is?

zero.