Math 2153 - Spring 2017 Name: $\boxed{\text{SOLUTION}}$

Quiz 8 - Take Home (10 pts) Recitation Time: ______________

SHOW ALL WORK!!! Unsupported answers might not receive full credit. Furthermore, please give me EXACT answers.

Problem 1 [4.5 pts] Suppose that you're a baker preparing cupcakes for Saint Patrick's Day. On top of each cupcake, you wish to put a four leafed clover (no stem) made of green icing. Suppose that in the xy-plane the equation of the shamrock is $r = \sin(2\theta) + \frac{1}{4}\sin(6\theta)$, where the center of the circular top is the origin and units of distance are measured in inches. Suppose you want this four leafed clover to be $\frac{1}{8}$ inch tall, and you want to prepare 200 cupcakes.

(a). [1.5 pts] How much icing by volume (in in^3) is required for the whole cupcake production (all 200)? (Hint: the trig identity $\sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$ as well as the double angle formula will be helpful.)

$$V = 200\int_0^{1/8}\int_0^{2\pi}\int_0^{\sin 2\theta + \frac{1}{4}\sin 6\theta} r\,dr\,d\theta\,dz = \frac{200}{8}\int_0^{2\pi}\frac{1}{2}\left(\sin 2\theta + \tfrac{1}{4}\sin 6\theta\right)^2 d\theta = \frac{25}{2}\int_0^{2\pi}\sin^2 2\theta + \tfrac{1}{2}\sin 2\theta\sin 6\theta + \tfrac{1}{16}\sin^2 6\theta\,d\theta$$

$$= \frac{25}{2}\int_0^{2\pi}\frac{1}{2} - \frac{\cos 4\theta}{2} + \frac{1}{4}\cos(4\theta) - \frac{1}{4}\cos(8\theta) + \frac{1}{32} - \frac{\cos(12\theta)}{32}\,d\theta$$

$$= \frac{25}{2}\left(\frac{17\theta}{32} - \frac{\sin 4\theta}{16} - \frac{\sin(8\theta)}{32} - \frac{\sin(12\theta)}{384}\right)\Bigg|_0^{2\pi} = \frac{25\cdot 17\pi}{32} = \boxed{\frac{425\pi}{32}}$$

vol of 1: $\xrightarrow{\div 200}$ $\frac{17\pi}{256}$

(b). [3 pts] Assume that the density of the icing is constant, i.e. $\rho(r,\theta,z) = c$. Given that $M_{\theta z} = 0$ (you need not do this computation), calculate the center of mass $(\bar{r},\bar{\theta},\bar{z})$ of one of the 0.125 inch thick shamrocks. You may just cite part (a) for part of the computation and use the formula (★) $\int x\cos ax\,dx = \frac{x\sin(ax)}{a} + \frac{\cos(ax)}{a^2}$ without proof. Also note $32\times 12 = 384$ and $144\times 32 = 4608$.

Constant density $\Rightarrow$ m = mass = density · volume = $\frac{17\pi c}{256}$.

Note: $I := \int_0^{2\pi}\int_0^{\sin(2\theta)+\frac{1}{4}\sin(6\theta)} r\,dr\,d\theta = \frac{17\pi}{256}\cdot 8 = \frac{17\pi}{32}$, since $I\cdot\int_0^{1/8} dz = \text{vol (1 shamrock)} = \frac{17\pi}{256}$

Then $M_{r\theta} = \int_0^{1/8}\int_0^{2\pi}\int_0^{\sin 2\theta + \frac{1}{4}\sin 6\theta} rz\cdot c\,dr\,d\theta\,dz = c\cdot\int_0^{1/8} z\,dz\cdot I = c\cdot\frac{1}{\underset{16\cdot 8}{128}}\cdot\frac{17\pi}{32}$, So $\boxed{\bar{z} = \frac{M_{r\theta}}{m} = \frac{\left(\frac{17\pi c}{16\cdot 16\cdot 16}\right)}{\left(\frac{17\pi c}{256}\right)} = \boxed{\frac{1}{16}}}$

$$M_{rz} = \int_0^{1/8}\int_0^{2\pi}\int_0^{\sin 2\theta + \frac{1}{4}\sin 6\theta} r\theta\cdot c\,dr\,d\theta\,dz = \frac{c}{16}\int_0^{2\pi}\theta\sin^2 2\theta + \tfrac{1}{2}\theta\sin 6\theta\sin 2\theta + \tfrac{\theta}{16}\sin^2 6\theta\,d\theta$$

$$= \frac{c}{16}\int_0^{2\pi}\frac{\theta}{2} - \frac{\theta\cos 4\theta}{2} + \frac{\theta}{4}\cos(4\theta) - \frac{\theta}{4}\cos(8\theta) + \frac{\theta}{32} - \frac{\theta\cos(12\theta)}{32}\,d\theta = \frac{c}{16}\int_0^{2\pi}\frac{17\theta}{32} - \frac{\theta\cos(4\theta)}{4} - \frac{\theta\cos(8\theta)}{4} - \frac{\theta\cos(12\theta)}{32}\,d\theta$$

use ★ in part

$$= \frac{c}{16}\left[\frac{17\theta^2}{64} - \frac{\theta\sin 4\theta}{16} - \frac{\cos(4\theta)}{64} + \frac{\theta\sin(8\theta)}{32} + \frac{\cos(8\theta)}{256} - \frac{\theta\sin(12\theta)}{384} - \frac{\cos(12\theta)}{4608}\right]\Bigg|_0^{2\pi} = \frac{17\pi^2 c}{256} \Rightarrow \boxed{\bar{\theta} = \frac{M_{rz}}{m} = \pi}$$

$M_{\theta z} = 0$, so $\bar{r} = \frac{M_{\theta z}}{m} = 0$. So $\boxed{(\bar{r},\bar{\theta},\bar{z}) = (0,\pi,\tfrac{1}{16})}$

Note: If you were given the proportion of powdered sugar needed (say it's half of the icing by volume) and the density of powdered sugar (it's 561 kg/m^3, or .0202 lb/in^3), you can easily calculate just how much you sugar you need.

Problem 2 [5.5 pts] Consider the solid region D bounded by the sphere $\rho = 4\cos\varphi$ and the hemisphere $\rho = 2$, $z \geq 0$.

(a). [1 pt] Convert the equations for both of these surfaces to Cartesian (x, y, z) coordinates.

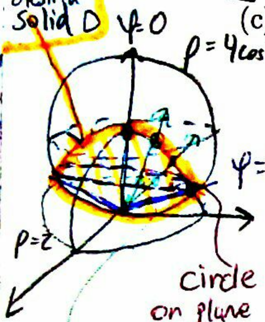
$$\rho = 4\cos\varphi \Rightarrow \rho^2 = 4(\rho\cos\varphi) \Rightarrow x^2+y^2+z^2 = 4z \Rightarrow x^2+y^2+z^2-4z=0 \Rightarrow x^2+y^2+z^2-4z+4=4 \Rightarrow \boxed{x^2+y^2+(z-2)^2=4}$$

$$(\rho=2)^2 \Rightarrow \rho^2 = 4 \Rightarrow \boxed{x^2+y^2+z^2=4}$$

(b). [1 pt] These two surfaces intersect in a circle, which itself is an intersection of (1) a cylinder parallel to the z-axis generated by that circle and (2) a plane parallel to xy-plane. Express the equations for both of these in spherical coordinates ("$\rho =$" form). (Hint for (1): using cylindrical coordinates may be a good intermediate step).

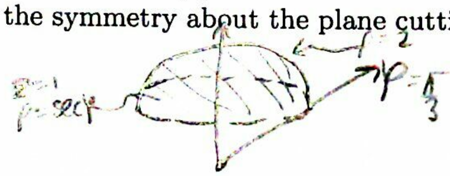
Notice on circle of intersection C that $4\cos\varphi = 2$, so $\cos\varphi = \frac{1}{2}$, so $\varphi = \frac{\pi}{3}$, $\rho = 2$, & $\rho = 4\cos\varphi$.

At pts on circle, $r = \rho\sin\varphi = 2\cdot\sin(\frac{\pi}{3}) = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$, so $r=\sqrt{3}$ is our cylinder. But $r = \rho\sin\varphi$, so $\rho\sin\varphi = \sqrt{3}$, and $\boxed{\rho = \sqrt{3}\csc\varphi}$ is our cylinder. As for the plane, on C, $z = \rho\cos\varphi = 2\cdot\cos(\frac{\pi}{3}) = 1$, so $\rho\cos\varphi = 1$, so $\boxed{\rho = \sec\varphi}$ is our plane.



first $\rho=\sec\varphi$ then $\rho=2$

(c). [2.5 pts] In part (b) you found along the way the φ value that all points on the circle of intersection satisfy. Use this to find the volume of the upper half of D. Then, use this to find the volume of the total solid D (take the symmetry about the plane cutting the solid in half for granted; no proof necessary).



$$2\left(\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\varphi}^{2} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta\right)$$

$$= 4\pi\int_0^{\frac{\pi}{3}}\left(\frac{8}{3}\sin\varphi - \frac{1}{3}\sec^3\varphi\sin\varphi\right)d\varphi = \frac{4\pi}{3}\left(-8\cos\varphi - \frac{1}{2}\tan^2\varphi\right)\Big|_0^{\frac{\pi}{3}} = \frac{4\pi}{3}\left(-4-\frac{1}{2}(3)+8\right) = \boxed{\frac{10\pi}{3}}$$

($\tan\varphi\sec^2\varphi$)

(d). [2 pts] Write the volume of the solid as a sum of two integrals: one computing the volume of an "ice cream cone" (with "ice cream") and one computing the volume of the part of a sphere with the aforementioned cone cut out. Then, compute that volume. This answer should match (c).

$$\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{2} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta + \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{4\cos\varphi} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \frac{16\pi}{3}\cdot(-\cos\varphi)\Big|_0^{\frac{\pi}{3}} + \frac{2\pi}{3}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 64\cos^3\varphi\sin\varphi\, d\varphi$$

$$= \frac{8\pi}{3} - \frac{128\pi}{3}\left[\frac{1}{4}\cos^4\varphi\right]\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{8\pi}{3} + \frac{2\pi}{3} = \boxed{\frac{10\pi}{3}}$$