Math 2153 - Spring 2017 Name: ______________________

Quiz 6 - SOLUTIONS Recitation Time: ______________________

SHOW ALL WORK!!! Unsupported answers might not receive full credit. Furthermore, give me EXACT answers (do NOT use your decimals in your final answers, though they may be used to approximate where a number is in a graph).

Problem 1 [2 pts] Express the volume of the solid bounded between the surface $z = xy^2\cos(xy^3)$ and the rectangle $R = \{(x,y) : 0 \le x \le \frac{\pi}{2}, 0 \le y \le 1\}$ in the xy-plane as an integral. Then, evaluate that integral.

Notice $z = \frac{\partial}{\partial y}(\frac{1}{3}\sin(xy^2))$, so

$$\iint_R z\,dA = \int_0^{\pi/2}\int_0^1 \frac{\partial}{\partial y}(\frac{1}{3}\sin(xy^3))dydx = \int_0^{\pi/2}\frac{1}{3}\sin(x)dx = (-\frac{1}{3}\cos x)\Big|_0^{\pi/2} = 0 + 1/3 = 1/3.$$

Problem 2 [3 pts] In this problem, we'll find the closest point(s) on the elliptic cone $4x^2 + 9y^2 - 16z^2 = 0$ to the point $(1,1,0)$.

(a). [0.5 pts] Set up the three equations resulting from using the method of Lagrange multipliers for the above scenario.

We're trying to minimize $D(x,y,z) = (x-1)^2 + y - 1)^2 + z^2$ subject to the constraint $g(x,y,z) = 4x^2 + 9y^2 - 16z^2 = 0$, so we're getting 3 equations from $\nabla D = \lambda \nabla g$, which are $2(x-1) = 8\lambda x$, $2(y-1) = 18\lambda y$, and $2z = -32\lambda z$.

(b). [1.5 pts] Use one of the above equations and the zero product property from high school algebra to find a potential value for λ and use that to solve for x, y, and z. Also, consider the other case that comes from that application of the zero-product property and find what point on the cone arises from it.

From the third equation in (a), $2z + 32\lambda z = 0$, so $2z(1 + 16\lambda) = 0$, meaning by the zero product property $(1) z = 0$ or $(2) 1 + 16\lambda = 0$. If $z = 0$, then $4x^2 + 9y^2 = 0$, meaning $x = y = 0$; this pair cannot satisfy the other 2 equations in (a), so we disregard this case. In case (2), $\lambda = -\frac{1}{16}$, so $2x - 2 = -\frac{1}{2}x$ and $2y - 2 = -\frac{9}{8}y$, meaning $x = \frac{4}{5}$ and $y = \frac{16}{25}$. Plugging these into the constraint $g(x,y,z) = 0$, we get $4/25 + (9 \cdot 16)/(25^2) = 244/625 = z^2$, so $z = \pm\frac{2\sqrt{61}}{25}$.

(c). [1 pt] Use (b) to find the closest point(s) on the elliptic cone $4x^2 + 9y^2 - 16z^2 = 0$ to the point $(1,1,0)$. Justify how you know that the point you found is indeed the closest point. You are encouraged to specifically cite theorems on the recitation handouts and appeal to basic geometric intuition about the cone.

$D(\frac{4}{5}, \frac{16}{25}, \pm\frac{2\sqrt{61}}{25}) = 25/625 + 81/625 + 244/625 = 350/625 = 14/25$. The method of Lagrange multipliers (Theorem 1.2, Procedure 1.3 on the handout) tells us that the maxima and minima of D (and hence the distance function) on the constraint surface occur at the points found through Lagrange multipliers. The cone is infinite, so no maximum distance occurs. Hence, the points found must be the closest points on the cone to $(1,1,0)$, as it is where the minimal value of D found through Lagrange mulipliers on the cone occurs, and hence it is the absolute minimum of D on the cone.

Problem 3 [5 pts] In this problem, we'll find the absolute maximum and absolute minimum values of the function $f(x, y) = 8xy$ in/on the region $R = \{(x, y) : 4x^2 + 9y^2 \leq 36\}$.

(a). [1 pt] Find the critical point(s) of f in the interior of R, classify it/them with the 2nd Derivative Test, and evaluate f at it/them.

$f_x = 8y$ and $f_y = 8x$, so the only critical point is $(0, 0)$, where $f_x = f_y = 0$. Also, for all points $f_{xx} = 0$, $f_{yy} = 0$, and $f_{xy} = 8$, so $D(0, 0) = -64 < 0$, so $(0, 0)$ is a saddle point. Also, $f(0, 0) = 0$.

(b). [0.5 pts] Set up the pair of equations obtained from using Lagrange multipliers on the boundary.

The boundary curve is $g(x, y) = 4x^2 + 9y^2 = 36$, and we're optimizing $f(x, y) = 8xy$ on it, so $\nabla f = \lambda \nabla g$ yields $8y = 8\lambda x$ and $8x = 18\lambda y$.

(c). [0.5 pts] Use part (b) to eliminate λ, and then use this elimination to obtain an equation $p(x, y) = 0$ where $p(x, y)$ is a polynomial.

A common multiple for both right hand sides is $72\lambda xy$, so multiplying the first equation by $9y$ and the second by $4x$, we obtain $72y^2 = 72\lambda xy = 32x^2$, so $72y^2 - 32x^2 = 0$, meaning $9y^2 - 4x^2 = 0$.

(d). [0.5 pts] Factor the above polynomial and use the zero product property from high school algebra to determine two possible SIMPLE relations between x and y.

$(3y - 2x)(3y + 2x) = 0$, so $y = \pm\frac{2}{3}x$.

(e). [1.5 pts] Plug the relations found in part (d) into the constraint (boundary curve) to find the pairs (x, y) on the boundary curve where extreme values occur (this comes for free from Lagrange Multipliers). Then, evaluate f at those points.

We have $9y^2 = 4x^2$, so plugging into the constraint curve $4x^2 + 9y^2 = 36$, we get $8x^2 = 36$, so $x = \pm\frac{3}{\sqrt{2}}$, so $y = \pm\sqrt{2}$. So, we have four points we're considering. Evaluating, we get $f(\frac{3}{\sqrt{2}}, \sqrt{2}) = f(-\frac{3}{\sqrt{2}}, -\sqrt{2}) = 24$ and $f(-\frac{3}{\sqrt{2}}, \sqrt{2}) = f(\frac{3}{\sqrt{2}}, -\sqrt{2}) = -24$.

(f). [1 pt] Use the information from parts (a) and (e) and two theorems and/or procedures from the recitation handouts (state which theorems they are and WHY they apply) to determine the absolute minimum and maximum values of f in/on R.

R is closed and bounded in $\mathbb{R}^2$ and f is continuous on R, so by the Strengthened Extreme Value Theorem, absolute extreme values for f in/on R occur either at critical points in the interior or on the boundary. The method of Lagrange multipliers applies to the boundary, telling us that maximum and minimum values on the boundary are found through Lagrange multipliers. So, the critical point in the interior is (0,0) and $f(0, 0) = 0$, the minimum value of f on the boundary is -24, and the maximum value of f on the boundary is 24. Therefore, by the Strengthened Extreme Value Theorem, the absolute maximum value of f in/on R is 24, and the absolute minimum value of f in/on R is -24.