Math 2153 - Spring 2017 Name: ______________________

Quiz 4 - SOLUTIONS Recitation Time: ______________________

SHOW ALL WORK!!! Unsupported answers might not receive full credit. Furthermore, give me EXACT answers (do NOT use your decimals in your final answers, though they may be used to approximate where a number is in a graph).

Problem 1 [2 pt] For the function $h(u,v) = \left(\frac{uv}{u-v}\right)^{3/2}$, calculate the two first partial derivatives h_u and h_v.

By the Chain Rule and then the Quotient Rule,

$$h_u = \frac{3}{2}\left(\frac{uv}{u-v}\right)^{1/2} \cdot \frac{\partial}{\partial u}\left(\frac{uv}{u-v}\right) = \frac{3}{2}\left(\frac{uv}{u-v}\right)^{1/2} \cdot \frac{(u-v)v - uv(1)}{(u-v)^2} = -\frac{3}{2}\frac{u^{1/2}v^{5/2}}{(u-v)^{5/2}}.$$

By very similar reasoning,

$$h_v = \frac{3}{2}\left(\frac{uv}{u-v}\right)^{1/2} \cdot \frac{\partial}{\partial v}\left(\frac{uv}{u-v}\right) = \frac{3}{2}\left(\frac{uv}{u-v}\right)^{1/2} \cdot \frac{(u-v)u - uv(-1)}{(u-v)^2} = \frac{3}{2}\frac{u^{5/2}v^{1/2}}{(u-v)^{5/2}}.$$

Problem 2 [3 pts] Consider the function $G(x,y) = \sin^{-1}(y - x^2)$.

(a). [0.5 pts] Find the domain of G.

The domain of $\sin^{-1}$ is $[-1,1]$, so we need $-1 \le y - x^2 \le 1$, and by adding x^2 to all parts of the sequence of inequalities, we get $x^2 - 1 \le y \le x^2 + 1$.

(b). [0.5 pts] Sketch the domain of G as a region in the xy-plane.

This is the region in the xy-plane between and including the parabolas $y = x^2 - 1$ and $y = x^2 + 1$, which are parallel and thus never intersect.

(c). [1 pt] State whether the above region is open, closed, or neither in the xy-plane and how you know. Include a description of what the interior and boundary points of the region are in your answer.

The interior points are all points strictly between the parabolas (you can draw a disc around these points that's small enough to still not intersect either parabola); in particular, they're not on either one. The boundary points are all points on either of the two parabolas (draw ANY disc around one of these points, and it intersects both the parabola and the area in between the parabolas). Since boundary points can't be interior points, the set is not open. Since all boundary points are included in our set, the set is closed.

(d). [1 pt] State whether the region is bounded or unbounded in the xy-plane and how you know.

Clearly, our set cannot be contained in a disc of finite size, since the parabolas themselves go off to infinity, meaning the points between them do too, so the set is unbounded.

Problem 3 [5 pts] Consider the function $f(x,y) = \frac{\sin(xy)}{x+y}$.

(a). [1 pt] Evaluate what this limit would be along the path $y = mx$ for any m using the rule $\displaystyle\lim_{x\to 0} \frac{\sin(g(x))}{g(x)} = 1$ if $g(x) \to 0$ as $x \to 0$.

If we set $y = mx$, $xy = mx^2$ and $x+y = (m+1)x$, so $\displaystyle\lim_{(x,mx)\to(0,0)} f(x,mx) = \lim_{x\to 0} \frac{\sin(mx^2)}{(m+1)x} \cdot \frac{mx}{mx} = \lim_{x\to 0} \frac{\sin(mx^2)}{mx^2} \cdot \frac{mx}{m+1} = \lim_{x\to 0}\left(\frac{\sin(mx^2)}{mx^2}\right) \cdot \lim_{x\to 0} \frac{mx}{m+1} = 1 \cdot 0 = 0.$

In the parts that follow, we will show that it is NOT enough to only consider the paths $y = mx$ to determine whether a limit exists, despite the fact they agree. Notice that as $x \to 0^+$, if we set $y = -\sin(x)$, then $y \to 0^-$ since $\sin(x) \to 0^+$ as $x \to 0^+$. So, this is yet another path to the origin as $x \to 0^+$. We'll show using Maclaurin series that the limit as (x, y) tends to $(0, 0)$ along this path is $-\infty$.

(b). [1 pt] Find the Maclaurin series for $-x\sin(x)$ and $x - \sin(x)$.

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$, so $-x\sin(x) = -x^2 + \frac{x^4}{3!} - \frac{x^6}{5!} + \cdots$ and $x - \sin(x) = \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$.

(c). [1 pt] Find the first 3 terms for the Maclaurin series for $\sin(-x\sin x)$.

We plug the series for $-x\sin(x)$ in for "x" into the series for $\sin(x)$:

$$\sin(-x\sin(x)) = \left(-x^2 + \frac{x^4}{3!} - \frac{x^6}{5!} + \cdots\right) - \frac{1}{3!}\left(-x^2 + \frac{x^4}{3!} - \cdots\right)^3 + \cdots$$

$$= -x^2 + \frac{x^4}{3!} + \left(\frac{1}{3!} - \frac{1}{5!}\right)x^6 + \cdots.$$

(d). [2 pts] Use parts (b) and (c) to show that as $x \to 0^+$ along the path $y = -\sin(x)$ $f(x,y)$ tends to $-\infty$. You are encouraged to just write the first 3 nonzero terms of each Maclaurin series and "$\cdots$" for subsequent terms (please DON'T write the full Σ expression). Using methods like in Example 3 on pages 91-93 in the textbook should be helpful in evaluating the resulting limit.

By parts (b) and (c), plugging in our Taylor series, we're looking at

$$\lim_{(x,-\sin(x))\to(0^+,0^-)} f(x,-\sin(x)) = \lim_{x\to 0^+} \frac{-x^2 + \frac{x^4}{3!} + \left(\frac{1}{3!} - \frac{1}{5!}\right)x^6 + \cdots}{\frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots}.$$

Treating this like an infinite degree polynomial, we employ the strategy for rational functions of a single variable: multiply top and bottom by $\frac{1}{x^2}$ to get:

$$\lim_{x\to 0^+} \frac{-1 + \frac{x^2}{3!} + \left(\frac{1}{3!} - \frac{1}{5!}\right)x^4 + \cdots}{\frac{x}{3!} - \frac{x^3}{5!} + \frac{x^5}{7!} + \cdots} = \frac{-1}{0^+} = -\infty.$$

Thus, our limit is $-\infty$, meaning as $x \to 0^+$ along the path $y = -\sin(x)$ $f(x,y)$ tends to $-\infty$.

Note: Notice how much easier the method in part (d) is than using L'Hospital's Rule. That requires 3 separate applications and UGLY Chain and Product Rule computations! You can generally get away with using this method instead of L'Hospital's Rule for evaluating a variety of limits.