Math 2153 - Spring 2017 Name: ______________________

Quiz 3 - SOLUTIONS Recitation Time: ______________________

SHOW ALL WORK!!! Unsupported answers might not receive full credit. Furthermore, give me EXACT answers (do NOT use your calculator), and do each computation completely. DO NOT leave anything in dot product or cross product form. You will lose points by doing so.

Problem 1 [9 pts] Consider the curve $\mathbf{r}(t) = \langle 5\cos t, 5\sin t, 12t\rangle$.

(a). [0.5 pts] Calculate the unit tangent vector $\mathbf{T}(t)$.

$\mathbf{r}'(t) = \langle -5\sin t, 5\cos t, 12\rangle$, so $|\mathbf{r}'(t)| = 13$ and
$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{13}\langle -5\sin t, 5\cos t, 12\rangle = \langle -\frac{5}{13}\sin t, \frac{5}{13}\cos t, \frac{12}{13}\rangle$.

(b). [0.5 pts] Calculate the unit normal vector $\mathbf{N}(t)$.

$\mathbf{T}'(t) = \langle -\frac{5}{13}\cos t, -\frac{5}{13}\sin t, 0\rangle$, so $|\mathbf{T}'(t)| = \frac{5}{13}$ and
$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\langle -\frac{5}{13}\cos t, -\frac{5}{13}\sin t, 0\rangle}{\frac{5}{13}} = \langle -\cos t, -\sin t, 0\rangle$.

(c). [0.5 pts] Compute the curvature κ at any time t.

$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\frac{5}{13}}{13} = \frac{5}{169}$. Notice that we already calculated $|\mathbf{T}'(t)|$ and $|\mathbf{r}'(t)|$ in parts (a) and (b).

(d). [0.5 pts] Calculate the unit binormal vector $\mathbf{B}(t)$.

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -(5/13)\sin t & (5/13)\cos t & 12/13 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \langle \tfrac{12}{13}\sin t, -\tfrac{12}{13}\cos t, \tfrac{5}{13}\rangle.$$

(e). [1 pt] Compute the torsion τ for any time t.

$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$, where $\frac{d\mathbf{B}}{ds} = \frac{1}{|\mathbf{r}'(t)|}\mathbf{B}'(t) = \frac{1}{13}\langle \frac{12}{13}\cos t, \frac{12}{13}\sin t, 0\rangle$, so

$$\tau = (-1)\langle \frac{12}{169}\cos t, \frac{12}{169}\sin t, 0\rangle \cdot \langle -\cos t, -\sin t, 0\rangle = \frac{12}{169}$$

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(f). [3 pts] Give the equations for the osculating planes for the curve at $t = 0$ and $t = \frac{\pi}{2}$ in $ax + by + cz = d$ form.

The general equation of a plane is $\mathbf{n}\cdot\langle x - x_0, y - y_0, z - z_0\rangle = 0$, where $\mathbf{n}$ is a vector normal to the plane and (x_0, y_0, z_0) is a point on the plane. The osculating plane at t_0 is tangent to the curve at the point $\mathbf{r}(t_0)$ and is spanned by $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$, meaning its normal vector $\mathbf{n}$ is $\mathbf{B}(t_0)$. Thus:

Since $\mathbf{r}(0) = \langle 5, 0, 0\rangle$ and $\mathbf{B}(0) = \langle 0, -\frac{12}{13}, \frac{5}{13}\rangle$, the equation of the osculating plane at $t = 0$ is $\langle 0, -\frac{12}{13}, \frac{5}{13}\rangle \cdot \langle x - 5, y - 0, z - 0\rangle$, i.e. $12y - 5z = 0$.

Since $\mathbf{r}(\frac{\pi}{2}) = \langle 0, 5, 6\pi\rangle$ and $\mathbf{B}(0) = \langle \frac{12}{13}, 0, \frac{5}{13}\rangle$, the equation of the osculating plane at $t = \frac{\pi}{2}$ is $\langle \frac{12}{13}, 0, \frac{5}{13}\rangle \cdot \langle x - 0, y - 5, z - 6\pi\rangle$, i.e. $12x + 5z = 30\pi$.

(g). [3 pts] Find the parametric description $\mathbf{R}(t)$ for the line of intersection of the two osculating planes found in part (f).

The general equation of a line $\mathbf{R}(t)$ is $\mathbf{R}(t) = \mathbf{a} + t\mathbf{v}$ where $\mathbf{a}$ is a point on the line and $\mathbf{v}$ is a vector in the direction of the line. So, our task is to find $\mathbf{a}$ and $\mathbf{v}$. Since $\mathbf{v}$ lies in both planes and $\mathbf{B}(t_0)$ is perpendicular to ALL vectors on the osculating plane at t_0 for any t_0, we have that $\mathbf{v}$ will be perpendicular to both $\mathbf{B}(0)$ and $\mathbf{B}(\frac{\pi}{2})$; thus, $\mathbf{v}$ points in the direction of $\mathbf{B}(0) \times \mathbf{B}(\frac{\pi}{2})$, or equivalently $13\mathbf{B}(0) \times 13\mathbf{B}(\frac{\pi}{2}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -12\cos t & 5 \\ 12 & 0 & 5 \end{vmatrix} = \langle -60, 60, 144 \rangle$. To find $\mathbf{a}$, we set one of the variables equal to 0 and solve for the other variables in the plane equations found in part (f). If we set $z = 0$, from $12x + 5z = 30\pi$, we get $x = \frac{5\pi}{2}$, and from $12y - 5z = 0$ we get $y = 0$. Hence, $\mathbf{a} = \langle \frac{5\pi}{2}, 0, 0 \rangle$ and

$$\mathbf{R}(t) = \langle \tfrac{5\pi}{2}, 0, 0 \rangle + t\langle -60, 60, 144 \rangle$$

Problem 2 [1 pt] Sketch four level curves for the function $z = e^{-x-y}$ in the same xy-plane. Label which z value corresponds to which curve on your graph. You are encouraged to NOT pick integer values for your level curves; pick numbers that make the graphing as easy as possible.

Since our function is $z = e^{-x-y}$, it makes the most sense to set $z = c$, where c is some power of e if we're going to graph level curves. Therefore, we pick (1) $z = e^0 = 1$ to get $0 = -x - y$ or equivalently $y = -x$; (2) $z = e^1$ to get $1 = -x - y$ or equivalently $y = -x - 1$; (3) $z = e^2$ to get $2 = -x - y$ or equivalently $y = -x - 2$; and (4) $z = e^{-1}$ to get $-1 = -x - y$ or equivalently $y = -x + 1$. These are very simple to graph, so I won't do so here.