Math 2153 - Spring 2017 Name: ______________________

Quiz 2 - SOLUTIONS Recitation Time: ______________________

SHOW ALL WORK!!! Unsupported answers might not receive full credit.

Problem 1 [0.5 pts] State the formula for $\frac{d}{dt}[f(t)\mathbf{r}(t)]$ where f is a scalar-valued function and $\mathbf{r}$ is a vector-valued function. Do not write $\mathbf{r}(t)$ as a pair of components $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ and distribute $f(t)$.

This is the product rule in your textbook, which is $\frac{d}{dt}[f(t)\mathbf{r}(t)] = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$.

Problem 2 [1.5 pts] Evaluate the limit $\lim_{t\to 0}\left(\frac{1-\cos t}{t}\mathbf{i} - \frac{e^{2t}-t-1}{t}\mathbf{j} + \frac{\cos t + t^2/2 - 1}{t^2}\mathbf{k}\right)$,

$\lim_{t\to 0}\left(\frac{1-\cos t}{t}\mathbf{i} - \frac{e^{2t}-t-1}{t}\mathbf{j} + \frac{\cos t + t^2/2 - 1}{t^2}\mathbf{k}\right) = \left(\lim_{t\to 0}\frac{1-\cos t}{t}\right)\mathbf{i} + \left(\lim_{t\to 0} -\frac{e^{2t}-t-1}{t}\right)\mathbf{j} + \left(\lim_{t\to 0}\frac{\cos t + t^2/2 - 1}{t^2}\right)\mathbf{k}$, so since $\lim_{t\to 0}\frac{1-\cos t}{t} = 0$, as you learned in first semester calculus, $\lim_{t\to 0}\frac{e^{2t}-t-1}{t} = \frac{d}{dt}(e^2t - t)|_{t=0} = 1$, and by applying L'Hospital's Rule twice (or, by using the Maclaurin series for $\cos t$) we have $\lim_{t\to 0}\frac{\cos t + t^2/2 - 1}{t^2} = 0$, we have that the above limit is $\langle 0, -1, 0\rangle$.

Problem 3 [4 pts] Consider the curve $\mathbf{r}(t) = \langle 2\sin 2t, 3t, 2\cos 2t\rangle$ for $0 \le t \le 4$.

(a). [2 pts] Find a formula for the arc length for the arc length $s(t)$ for any t in general. (Warning: this is distinct from the length of the complete curve).

$\mathbf{r}'(t) = \langle 4\cos 2t, 3, -4\sin 2t\rangle$, so

$$|\mathbf{r}'(t)| = \sqrt{(4\cos^2 2t)^2 + 3^2 + (-4\sin 2t)^2} = \sqrt{16\cos^2 2t + 9 + 16\sin^2 2t} = \sqrt{16+9} = 5.$$

Thus

$$s(t) = \int_0^t |\mathbf{r}'(u)|du = \int_0^t 5du = (5u)|_{u=0}^t = 5t.$$

(b). [2 pts] State whether the curve is parametrized by arc length and how you know [0.5 pts]. If it is not parametrized by arc length, find another description of the curve that uses arc length as a parameter [1.5 pts].

Since $|\mathbf{r}'(t)| = 5 \neq 1$, the curve is not parametrized by arc length. To parametrize by arc length, we use the expression for s we found in part (a), $s = 5t$, and solve for t in terms of s, i.e. $t = \frac{s}{5}$. Then, we substitute that expression into $\mathbf{r}(t)$ to make $\mathbf{r}$ a function of s. Thus the description of $\mathbf{r}$ that uses arc length s as a parameter is

$$\mathbf{r}(s) = \langle 2\sin\frac{2s}{5}, \frac{3s}{5}, 2\cos\frac{2s}{5}\rangle.$$

Problem 4 [4 pts] Suppose that the acceleration of a projectile is given by the vector-valued function $\mathbf{a}(t) = \langle te^t, \sin^2 t, \frac{1}{t+1}\rangle$ for $t \geq 0$, and suppose the initial position and velocity are given by $\mathbf{r}(0) = \langle 2, \frac{1}{2}, 1\rangle$ and $\mathbf{v}(0) = \langle 2, 2, 2, \rangle$, respectively. Find the value of the position function $\mathbf{r}$ for all values of $t \geq 0$, i.e. find $\mathbf{r}(t)$. Show your work. Citing an online integral calculator or a table of integrals will award you no credit for the evaluation of the integral(s).

$$\mathbf{v}(t) = \int \mathbf{a}(t)dt = \langle \int te^t dt, \int \sin^2 t dt, \int \frac{1}{t+1}\rangle = \langle te^t - e^t + c_1, \frac{t}{2} - \frac{\sin 2t}{4} + c_2, \ln(t+1) + c_3\rangle,$$

where the integrals are solved via integration by parts, the half angle formula $\sin^2 t = \frac{1-\cos 2t}{2}$, and u-substitution with $u = t+1$, respectively. To solve for c_1, c_2, and c_3, we use our initial condition for $\mathbf{v}$:

$$\mathbf{v}(0) = \langle -1 + c_1, c_2, c_3\rangle = \langle 2, 2, 2\rangle,$$

meaning $c_1 = 3$ and $c_2 = c_3 = 2$. Therefore,

$$\mathbf{v}(t) = \langle te^t - e^t + 3, \frac{t}{2} - \frac{\sin 2t}{4} + 2, \ln(t+1) + 2\rangle.$$

By integration by parts setting $u = \ln(t+1)$ and $dv = dx$,

$$\int \ln(t+1) = t\ln(t+1) - \int \frac{t}{t+1} dt = t\ln(t+1) - \int \left(1 - \frac{1}{t+1}\right) dt = t\ln(t+1) - t + \ln(t+1) + C,$$

so

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt = \langle te^t - 2e^t + 3t + d_1, \frac{t^2}{4} + \frac{\cos 2t}{8} + 2t + d_2, (t+1)\ln(t+1) - t + 2t + d_3\rangle.$$

$\mathbf{r}(0) = \langle -2 + d_1, \frac{1}{8} + d_2, d_3\rangle = \langle 2, \frac{1}{2}, 1\rangle$, so $d_1 = 4$, $d_2 = \frac{3}{8}$, and $d_3 = 1$. Therefore,

$$\mathbf{r}(t) = \langle (t-2)e^t + 3t + 4, \frac{t^2}{4} + \frac{\cos 2t}{8} + 2t + \frac{3}{8}, (t+1)\ln(t+1) + t + 1\rangle.$$