Math 2153 - Spring 2017 Name: ______________________

Quiz 1 - SOLUTIONS Recitation Time: ______________________

SHOW ALL WORK!!! Unsupported answers might not receive full credit.

Problem 1 [2 pts] A model airplane is flying horizontally due north at 30 mi/hr when it encounters a horizontal crosswind blowing east at 20 mi/hr and an updraft blowing vertically upward at 15 mi/hr. Find the position vector that represents the velocity of the plane relative to the ground, and find the speed of the plane relative to the ground.
Per the fomula given on the 12.1-12.3 handout, $\mathbf{v}_g = \mathbf{v}_a + \mathbf{w} + \mathbf{d}$, where velocity in calm air is given by $\mathbf{v}_a$, velocity relative to the ground is given by $\mathbf{v}_g$, $\mathbf{w}$ is the velocity of the crosswind, and $\mathbf{d}$ is the velocity of the updraft (or downdraft, depending on the problem). We abide by the convention that north is the positive y direction, east is the positive x direction, and up is the positive z direction. By the prompt, we have $\mathbf{v}_a = \langle 0, 30, 0\rangle$, $\mathbf{v}_g$ is what we're tasked to find, $\mathbf{w} = \langle 20, 0, 0\rangle$, and $\mathbf{d} = \langle 0, 0, 15\rangle$. Therefore,

$$\mathbf{v}_g = \mathbf{v}_a + \mathbf{w} + \mathbf{d} = \langle 20, 30, 15\rangle,$$

and the speed relative to the ground is the magnitude of this vector, which is $\sqrt{400+900+225} = \sqrt{1525}$ mi/hr.

Problem 2 [3 pts] Consider the points $A(5,5,2)$, $B(9,11,4)$, and $C(3,2,1)$. Use the cross product to determine whether the three points are colinear and explain how you know this from your answer. (Hint: How are colinearity of vectors and "parallelness" of vectors related?)
Since we are tasked with using the cross product, we should consider 2 vectors that we wish to show are on the same line, which will show that A, B, and C are on the same line. For this, we pick the vectors $\overrightarrow{AB} = \langle 9-5, 11-5, 4-2\rangle = \langle 4, 6, 2\rangle$ and $\overrightarrow{AC} = \langle 3-5, 5-2, 1-2\rangle = \langle -2, -3, -1\rangle$, and we consider their cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 6 & 2 \\ -2 & -3 & -1 \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ -3 & -1 \end{vmatrix}\mathbf{i} + \begin{vmatrix} 4 & 2 \\ -2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 6 & 2 \\ -3 & -1 \end{vmatrix}\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

By Theorem 12.3 in the textbook, a cross product of 2 vectors is $\mathbf{0}$ if and only if the vectors are parallel. Therefore, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel vectors, meaning they are scalar multiples of each other. However, because these vectors share a common starting point A, this means that the vectors themselves are collinear. Consequently, the points on these vectors, namely A, B, and C, are all collinear.

Problem 3 [5 pts] Consider the point $P(-5,7)$ and the line ℓ given by $y=5x$.

(a). [0.5 pts] Find any vector $\mathbf{v}$ in the direction of ℓ.

A vector in the direction of ℓ would be a vector starting at the origin $(0,0)$ ending at a point on the line, like $(1,5)$. So, $\mathbf{v}=\langle 1,5\rangle$ works.

(b). [0.5 pts] Find the position vector $\mathbf{u}$ corresponding to P. A position vector starts at the origin and ends at the desired point, so $\mathbf{u}=\langle -5,7\rangle$.

(c). [1 pt] Find $\text{proj}_{\mathbf{v}}\mathbf{u}$.

$proj_{\mathbf{v}}\mathbf{u}=\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}=\frac{\langle -5,7\rangle\cdot\langle 1,5\rangle}{\langle 1,5\rangle\cdot\langle 1,5\rangle}\langle 1,5\rangle=\frac{(-5)\cdot 1+7\cdot 5}{1\cdot 1+5\cdot 5}\langle 1,5\rangle=\frac{30}{26}\langle 1,5\rangle=\frac{15}{13}\langle 1,5\rangle=\langle\frac{15}{13},\frac{75}{13}\rangle.$

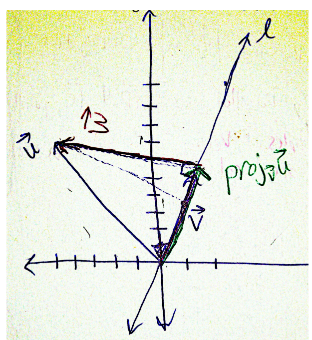
(d). [1 pt] Show that $\mathbf{w}=\mathbf{u}-\text{proj}_{\mathbf{v}}\mathbf{u}$ is perpendicular to $\mathbf{v}$.

$\mathbf{w}=\langle -5,7\rangle-\langle\frac{15}{13},\frac{75}{13}\rangle=\langle -\frac{80}{13},\frac{16}{13}\rangle$, so

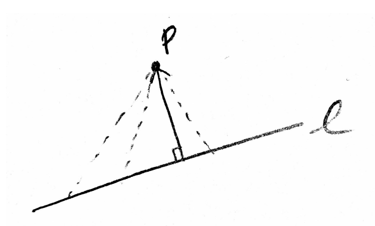
$\mathbf{w}\cdot\mathbf{v}=\langle -\frac{80}{13},\frac{16}{13}\rangle\cdot\langle 1,5\rangle=0$. Since the dot product of 2 vectors is 0 if and only if the vectors are parallel, it follows that $\mathbf{w}$ is perpendicular to $\mathbf{v}$.

(e). [2 pts] Use a picture with the above 4 vectors and a fact about side lengths in right triangles to explain why $|\mathbf{w}|$ is the least distance between P and ℓ. Consider what other "distances" would look like in your answer.

Here is a picture with all 4 vectors:



However, to make the main ideas at play as simple as possible, we'll look at the following picture instead: A "distance" is the length of a segment between 2 points,



so we're considering lengths of segments from P to points on ℓ. $|\mathbf{w}|$ is the length of the arrow in the above the paper, perpendicular to ℓ. Any other "distance" than $|\mathbf{w}|$ (some examples are dotted in the above picture) would necessarily be the hypotenuse of a right triangle with one leg on ℓ and the other leg given by $|\mathbf{w}|$. Since hypotenuses are ALWAYS longer than legs, any other "distance" would have to be greater than $|\mathbf{w}|$, meaning $|\mathbf{w}|$ is the least distance from P to ℓ.