Doctoral thesis proposal: A classification of integrally closed rings between $\mathbb{Z}_p[X_1, ..., X_n]$ and $\mathbb{Q}[X_1, ..., X_n]$

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1 Introduction

As our setup, let p be a prime positive integer. Consider the valuation ring $\mathbb{Z}_p \subseteq \mathbb{Q}$ with corresponding valuation v_p on $\mathbb{Q}$ normalized such that $v_p(p) = 1$. Over the course of this paper, we will be examining the structure of integrally closed rings D between $\mathbb{Z}_p[X_1, ..., X_n]$ and $\mathbb{Q}[X_1, ..., X_n]$, establishing what is known and what has yet to be discovered. In [9], Loper and Tartarone gave an essentially complete classification of all such rings D in the case $n = 1$ by examining trees of valuation overrings of D subject to a special "Maclane ordering," constructing these valuation overrings inductively with "key polynomials" as Saunders Maclane did in [10]. When we say S is an **overring** of R, we mean that $R \subseteq S$ and R and S have the same quotient field. However, the case of several variables has not been studied, and the goal of the forthcoming dissertation which this paper serves as a proposal for is to establish results about the structure and construction (via valuation overrings) of the integrally closed domains between $\mathbb{Z}_p[X_1, ..., X_n]$ and $\mathbb{Q}[X_1, ..., X_n]$. The foundation for this work and much of the work ahead is a theorem due to Krull, which we will now state.

Theorem 1.1 [Krull] *Let D be an integral domain which is integrally closed in its quotient field K. Then, D is equal to the intersection if its valuation overrings.*

Therefore, as a start, we know an integrally closed domain is the intersection of its valuation overrings, and since the integrally closed domain lies in $\mathbb{Q}[X_1, ..., X_n]$, we may include it in our intersection. However, as far as determining the structure of the given domain goes, this intersection (whether it's with or without $\mathbb{Q}[X_1, ..., X_n]$) isn't very helpful. Ideally, we would be able to represent the domain as the intersection of a strictly smaller collection with $\mathbb{Q}[X_1, ..., X_n]$, and perhaps if the collection is "nice" further statements can be made about the structure of that integrally closed domain. To start, we notice that if V and W are in this intersection and $V \cap \mathbb{Q}[X_1, ..., X_n] \subseteq W \cap \mathbb{Q}[X_1, ..., X_n]$, then we may omit W from this intersection. This suggests an ordering on the valuation rings contained in $\mathbb{Q}(X_1, ..., X_n)$ from which the minimal elements are the valuations we use to construct integrally closed rings. In the case of 1 variable, with a slight modification to this rough definition, an ordering on the valuation domains was found (the Maclane ordering $\preceq_{Mac}$, which will be discussed later), and this ordering allowed for the construction of a locally finite tree whose properties determined the structure of the integrally closed domain which could be represented as an intersection of valuation domains in that tree. Moreover, this tree almost completely determines the local structure of the given integrally closed domain, by a theorem we state loosely now and then more precisely later in this paper.

Theorem 1.2 *[9, Theorem 3.2] Suppose D is an integrally closed domain between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$, $\mathcal{V}$ is a collection of valuation domains with some hypotheses (to be discussed) that can be represented as the maximal vertices of a locally finite tree, and $D = \cap_{V \in \mathcal{V}} D_V$, where $D_V := V \cap \mathbb{Q}[X]$ for any valuation ring V. Then,*

(a) *If P is a prime ideal of D containing p, then $D_P = (D_V)_Q$ for some $V \in \mathcal{V}$ and some prime ideal Q in D_V containing p. Moreover, there is a valuation ring (R, M) in $\mathbb{Q}(X)$ containing D and p such that $D_P = (D_R)_{M \cap D_R}$.*

(b) *Let (W, M_W) be a valuation overring of D minimal with respect to $\preceq_{Mac}$, and let $P = M_W \cap D$. Then, for some prime ideal Q in D_W containing p, $D_P = (D_W)_Q$ and $D_W = D_P \cap \mathbb{Q}[X]$.*

(c) *Each $V \in \mathcal{V}$ contains p and is minimal under $\preceq_{Mac}$.*

Unfortunately, in the case of several variables, while the given "Maclane ordering" can be extended naturally, the nice results that follow from it in one variable can't really be salvaged. Fortunately, this ordering is equivalent to an inductive construction of valuations, so in the pursuit of a smaller and more insightful collection of valuation overrings that represent the given integrally closed domain, we will therefore be studying various valuations on $\mathbb{Q}(X_1, ..., X_n)$.

In the single variable case, results were found by extending v_p to $\mathbb{Q}(X)$ in various ways to obtain a classification of all possible extensions of v_p to $\mathbb{Q}(X)$ and, consequently, a classification of all integrally closed overrings of $\mathbb{Z}_p[X]$ contained in $\mathbb{Q}[X]$. We will discuss these methods in detail in the first few sections. The remaining sections of this paper will attempt to determine what can and can't be salvaged for integrally closed domains between $\mathbb{Z}_p[X_1, ..., X_n]$ and $\mathbb{Q}[X_1, ..., X_n]$ from these methods, find new methods to study these rings, and pose questions and conjectures as to what may be true for these rings, in preparation for a dissertation which will address these matters more completely.

2 The case of one variable

In this section, we pull from [5] and [9]. As our setup, let p be a prime positive integer. Consider the valuation ring $\mathbb{Z}_p \subseteq \mathbb{Q}$ with corresponding valuation v_p on $\mathbb{Q}$ normalized such that $v_p(p) = 1$. Over the course of this section, we will be extending v_p to $\mathbb{Q}(X)$ in various ways to obtain a classification of all possible extensions of v_p to $\mathbb{Q}(X)$ and consequently, a classification of all integrally closed overrings of $\mathbb{Z}_p[X]$ contained in $\mathbb{Q}[X]$. However, in many cases, we will give more general definitions for this theory so that we can consider discrete valuations on $\mathbb{Q}(X)[Y]$, which will perhaps allow us to extend some of these classification results to polynomial rings in several variables (or find their analogues for them). We begin with some notation and a result on integrally closed domains with the simplest representation possible: one valuation ring intersected with $\mathbb{Q}[X]$.

Notation 2.1 *Let T_p denote the collection of valuation overrings of $\mathbb{Z}[X]$ containing p as a nonunit; these are called* **p-unitary.** *For any domain $D \supseteq \mathbb{Z}[X]$, its* **p-unitary ideals** *are those containing p.*

Lemma 2.2 [9, Lemma 1.3] *Let V be a valuation domain in T_p and let M_V be the maximal ideal of V. Let $D_V = V \cap \mathbb{Q}[X]$ and let $P_V = M_V \cap D_V$. Then, $\operatorname{rad}(pD_V) \in \operatorname{Spec}(D_V)$ and if the value group of V is a subgroup of $(\mathbb{Q}, +)$, then $(D_V)_{P_V} = V$.*

Definition 2.3 *Let v_0 be a discrete valuation on a field K. A* **first stage inductive valuation** *is an extension v_1 of v_0 to $K(X)$ which is defined by assigning a value $\mu_1 \in \mathbb{R}^+$ to X and assigning to an arbitrary polynomial $f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in K[X]$ the value*

$$v_1(f(X)) := \min_{i=0,\ldots,n} \{v_0(a_i) + v_1(X^i)\} = \min_{i=0,\ldots,n} \{v_0(a_i) + i\mu_1\}$$

The corresponding valuation domain for this valuation, denoted V_1, is a **first stage inductive valuation domain***. If $\mu_1 \in \mathbb{Q}$, we call V_1* **commensurable***; if not, we call V_1* **incommensurable***. Note that this terminology applies to k^{th} stage valuation domains (to be defined on the next page) as well.*

Definition 2.4 *Let V_1 be as in the immediately preceding definition, and suppose $K = \mathbb{Q}$ with $v_0 = v_p$. The domain $D_1 = V_1 \cap \mathbb{Q}[X]$ is called a* **first stage inductive polynomial domain***. The maximal ideal of V_1 contracts to a height-one prime P_1 of D_1 which we call the* **valuation prime** *of D_1.*

Lemma 2.5 [9, Lemma 1.1] *Let $\alpha > \beta$ be two positive rational numbers and $D_{1,\alpha}$ and $D_{1,\beta}$ be their first stage inductive polynomial domains. Then $D_{1,\alpha}$ is an overring of $D_{1,\beta}$.*

Definition 2.6 *A* **Krull domain** *is an integral domain D with quotient field K that may be represented as an intersection of discrete valuation overrings $\{V_\lambda\}$ of D contained in K where for each $x \in K \setminus \{0\}$, x is a unit in all but finitely many V_λ.*

Remark 2.7 *If $v_1(X) \in \mathbb{Q}$, D_1 is a Krull domain since $\mathbb{Q}[X]$ is (as a Noetherian integrally closed domain), V_1 is one of the DVR's involved in the locally finite intersection representation of D_1, and V_1 is the only such DVR in which p is a nonunit.*

2.1 The extension process in the commensurable case

As stated in the introduction of this section, we wish to construct all possible valuations of $\mathbb{Q}(X)$ extending the valuation v_p on $\mathbb{Q}$, and it turns out that using first stage inductive valuation domains and extending them using an inductive procedure (and iterating this procedure) yields these extensions. In this subsection, we describe this procedure, beginning with the fundamental tool for this procedure: key polynomials. For now, we focus on the commensurable case (where the extended values are rational). In the next subsection, we'll discuss the incommensurable case.

Definition 2.8 *Let D be an integral domain, v be a valuation on D, and $a, b \in D$. We say a and b are* **equivalent** *(denoted $a \sim b$ in v) if $v(a-b) > v(a) = v(b)$, and we say b is* **equivalence divisible** *by a (denoted $a|b$ in v) if there exists an element $c \in D$ such that $ac \sim b$ in v.*

Note that $\sim$ is a congruence relation with respect to multiplication.

Definition 2.9 *Let K be a field and v be a valuation on $K[X]$. Then, a polynomial $\phi(X) \in K[X]$ is a* **key polynomial** *over v if*

(i) *if $\phi(X)|f(X)g(X)$ in v, then either $\phi(X)|f(X)$ or $\phi(X)|g(X)$ in v; in this case, we say $\phi(X)$ is* **equivalence-irreducible***;*

(ii) *if $\phi(X)|h(X)$ in v and $h(X) \neq 0$, then $\deg \phi(X) \leq \deg h(X)$;*

(iii) *$\phi(X)$ is monic.*

If we're given an extension v_1 to $K(X)$ of a discrete valuation v_0 on a field K, then given a key polynomial $\phi_2(X)$ over v_1 and a real number μ_2 greater than $v_1(\phi_2(X))$, we may define a function v_2 on $K[X]$ by assigning $\phi_2(X)$ the value μ_2 and setting

$$v_2(h(X)) := \min_i \{v_1(h_i(X)) + i\mu_2\}$$

for an arbitrary polynomial $h(X) = \sum_i h_i(X)(\phi(X))^i$ (this is the expansion of h with respect to ϕ, i.e. h_i are chosen such that $\deg h_i < \deg \phi$ for all i).

Notation 2.10 *Notationally, this is summarized as $v_2 = [v_1, v_2\phi = \mu_2]$ and $v_1 < v_2$. We call v_2 an* **augmented valuation** *of v_1.*

Definition 2.11 *We can iterate this process with new key polynomials ϕ_i at each stage to obtain a* **series of augmented inductive valuations**

$$v_1 < v_2 < \cdots < v_{k-1} < v_k < \cdots$$

where at each stage $v_k = [v_{k-1}, v_k\phi_k = \mu_k]$, where $\deg \phi_{k-1} \leq \deg \phi_k$. We call v_k a **kth stage inductive valuation***, and it has corresponding* **kth stage inductive valuation domain** *V_k. For $j < k$, we say v_j is a* **descended valuation** *of v_k. If the μ_k are rational, then we call these valuations* **commensurable***; if not, they are* **incommensurable***.*

Definition 2.12 *Given an infinite series of augmented inductive valuations as above, we have for every nonzero polynomial $g(X)$ that $v_{i+1}g(X) \geq v_i g(X)$ for all $i \in \mathbb{N}$. Furthermore, if for each such g the sequence $v_1 g(X) < v_2 g(X) < \cdots$ does not tend to ∞, then the limit of this sequence is denoted $v_\infty g(X)$, and this defines a valuation v_∞ which we call the* **limit valuation** *of the sequence $\{v_i\}_{i=1}^{\infty}$. If v_∞ is finite on every $f \in K[X]$, we say v_∞ is a* **finite limit valuation***; otherwise, we say v_∞ is an* **infinite limit valuation***. Analogously, when $K = \mathbb{Q}$, we may construct valuation domains V_∞ and polynomial domains $D_\infty := V_\infty \cap \mathbb{Q}[X]$ with valuation prime P_∞.*

Before describing the structure of D_∞, let us recall the definition of a Prüfer domain.

Definition 2.13 *An integral domain D is a Prüfer domain if every finitely generated ideal of D is invertible or, equivalently, D_P is a valuation domain for each prime ideal P of D. Pictorially, an integral domain is Prüfer if its lattice of prime ideals forms a tree.*

Theorem 2.14 [9, Proposition 1.20, Corollaries 1.21 and 1.22] *A finite limit valuation domain V_∞ is one-dimensional, and its residue field is algebraic over $\mathbb{F}_p$ (though, it may be infinite). Also, D_∞ is a one-dimensional Prüfer domain with $P_\infty = \mathrm{rad}(pD_\infty)$ among its maximal ideals (in fact, $(D_\infty)_{P_\infty} = V_\infty$), and its valuation overrings are precisely V_∞ and the valuation overrings of $\mathbb{Q}[X]$.*

Theorem 2.15 [9, Lemma 1.23, Proposition 1.24, Corollaries 1.25 and 1.26] *If V_∞ is an infinite limit valuation domain, then its residue field is a finite field of order p^n for some $n \in \mathbb{N}$, and all polynomials with infinite value are multiples of a single irreducible $f \in \mathbb{Q}[X]$. Moreover, P_∞ is a height two maximal ideal and the radical of pD_∞, and D_∞ is a two-dimensional Prüfer domain whose valuation overrings are precisely V_∞ and the valuation overrings of $\mathbb{Q}[X]$.*

Theorem 2.16 [10, Theorem 8.1] *Let K be a field with a discrete valuation v_0. Then, every discrete valuation on $K[X]$ extending v_0 can be represented as either an augmented commensurable valuation or as a limit valuation of a series of augmented inductive commensurable valuations.*

2.2 The extension process in the incommensurable case

Let V_k be a k^{th} stage commensurable valuation domain, and consider $v_{k+1} = [v_k, v_{k+1}(\phi_{k+1}) = \mu_{k+1}]$, where μ_{k+1} is irrational. Let all notation (M_k, D_k, P_k for any $k \in \mathbb{N}$) be as before.

Theorem 2.17 [10, Theorem 14.2] *Given V_{k+1} as above, its residue field is a finite field F_{k+1}. Moreover, $D_{k+1}/P_{k+1} \cong F_{k+1}$, meaning P_{k+1} is a maximal ideal.*

Lemma 2.18 [9, Lemmas 1.15 and 1.16, Corollaries 1.17 and 1.18] *With all notation as in the theorem, P_{k+1} is the unique height two prime ideal of D_{k+1} and the radical of pD_{k+1}. Consequently, by Krull's Principal Ideal Theorem, D_{k+1} is not Noetherian. Moreover, $(D_{k+1})_{P_{k+1}} \neq V_{k+1}$, as V_{k+1} is one-dimensional by hypothesis.*

Thus, we have some significant contrast with the commensurable case.

2.3 Two-dimensional valuation rings in T_p

In the previous sections, we focused exclusively on the classification of one-dimensional valuation overrings of $\mathbb{Z}_p[X]$ (i.e. those with value group contained in the real numbers). However, there are two-dimensional valuation overrings as well (and this is the largest the dimension of valuation overrings go since $\mathbb{Z}_p[X]$ is Noetherian and hence **Jaffard**, i.e. the maximal Krull dimension of the valuation overrings coincides with the Krull dimension of $\mathbb{Z}_p[X]$, which is 2), and in this section, we describe their construction and classification.

A given two-dimensional valuation ring $W \in T_p$ has a one-dimensional valuation overring V, which is obtained by localization at W's height one prime ideal. We have two cases: $V \in T_p$ and $V \notin T_p$. In the latter case, we know $V = \mathbb{Q}[X]_{(f)}$ for some irreducible $f \in \mathbb{Q}[X]$, and these domains are infinite limit valuation domains. The former case has more nuance. In this case, we know V is either inductive commensurable, inductive incommensurable, or finite limit. However, the case that V is inductive incommensurable or finite limit are impossible since the residue field of such a V is algebraic over a finite field, and those fields admit no nontrivial valuations.

Definition 2.19 *An* **upside down domain** *is a two-dimensional valuation domain V such that all its valuation overrings are inductive commensurable domains.*

We can construct upside down domains with three different equivalent methods: (1) as pullbacks of inductive commensurable domains; (2) as ultrafilter limits of sequences of inductive commensurable domains; (3) as Maclane extensions of inductive commensurable domains with extended values lying outside the real numbers. Since these are proven to be equivalent, we will just describe the first of these methods in this paper.

Construction 2.20 (The pullback construction) *Let V be an upside-down valuation domain with maximal ideal M and height-one prime Q. V_Q is the commensurable valuation domain corresponding to V. Therefore, the residue field of V_Q is of the form $F(Y)$ where F is a finite extension of $\mathbb{F}_p$. This field contains the valuation domain V/Q, so $V/Q \cong F[Y]_{(f)}$ for some irreducible $f \in F[Y]$ or $V/Q \cong F[1/Y]_{(1/Y)}$ (these are the only valuation rings inside $F(Y)$). Consequently, the maximal ideal of V is principal since V/Q is a DVR. Note that in this construction, we construct V from V_Q: we find a DVR such as above in the residue field of V_Q and then pull back that DVR via the same mapping from V_Q to its residue field. Summed up diagrammatically, we have:*

$$\begin{array}{ccc} V & \longrightarrow & V/Q \cong \bar{V} \subseteq F(Y) \\ \downarrow & & \downarrow \\ V_Q & \longrightarrow & V_Q/QV_Q \cong F(Y) \end{array}$$

Thus, three cases occur.

Case 1: *$V/Q \cong F[Y]_{(\Psi)}$ where $\Psi(Y) \neq Y$ is irreducible.*

Case 2: *$V/Q \cong F[Y]_{(Y)}$.*

Case 3: *$V/Q \cong F[1/Y]_{(1/Y)}$.*

Definition 2.21 *Corresponding to the cases in these constructions, we define* **Case 1, Case 2, and Case 3 upside-down valuation domains**. *In all cases, we denote these domains (V_{usd}, M_{usd}) and set $D_{usd} = V_{usd} \cap \mathbb{Q}[X]$ (we call D_{usd} an* **upside down polynomial domain***) and $P_{usd} = M_{usd} \cap D_{usd}$ (we call P_{usd} the* **valuation prime***).*

Let's first focus on Cases 1 and 2, as these are simpler.

Proposition 2.22 [9, Propositions 1.30 and 1.31] *Since $D_{usd}/P_{usd} \subseteq V_{usd}/M_{usd}$ and V_{usd}/M_{usd} is a finite field, in Case 1 and Case 2, P_{usd} is a maximal ideal. Moreover, if $V = (V_{usd})_Q$ is the commensurable valuation overring of V_{usd}, then $D_{usd} = D_V$.*

Case 3 has more nuance.

Proposition 2.23 [9, Lemma 1.32 and Corollary 1.33] *If V_{usd} is a Case 3 upside down domain, P_{usd} is the unique height-two prime ideal of D_{usd}, the radical of pD_{usd}, and the only prime ideal of D_{usd} containing p, and therefore, D_{usd} is not Noetherian by Krull's Principal Ideal Theorem; moreover, D_{usd}/P_{usd} is finite.*

2.4 Maclane ordering of valuation domains

As stated in the introduction, we wish to find a more insightful representation of an integrally closed domain D than as the intersection of all its valuation overrings. If an intersection of a smaller set of representatives, all having particular nice properties, can be found, more can be said about the given integrally closed domain D. In the previous subsections, we constructed all valuation rings in $\mathbb{Q}(X)$ containing $\mathbb{Z}_p[X]$, so in this subsection, we wish to construct an ordering that makes sense on all these valuation rings we've constructed and use this ordering to define a "minimal" intersection of valuation overrings.

Definition 2.24 *Let $(V, M), (W, N) \in T_p$. We write $V \preceq_{Mac} W$ if $D_V \subseteq D_W$ and $M \cap D_V \subseteq N \cap D_W$.*

Proposition 2.25 [9, Proposition 2.5, Corollary 2.6] *Let V be inductive commensurable and W be either an inductive (either commensurable or incommensurable) or limit valuation domain, and suppose $V, W \in T_p$. Then, $V \preceq_{Mac} W$ if and only if W is an extension of V. Consequently, if $V \preceq_{Mac} W$ and $W \preceq_{Mac} V$, then $V = W$. Therefore, $\preceq_{Mac}$ defines an order relation on the set of Maclane valuation domains, i.e. the inductive and limit valuation domains. In fact, it's equivalent to the inductive extension process.*

More can be said about this ordering.

Theorem 2.26 [9, Theorem 2.8, Lemmas 2.10, 2.11, 2.12, 2.13]

(1) *Limit valuation domains are maximal under* $\preceq_{Mac}$.

(2) *If* V *is an inductive commensurable domain with corresponding Case 1 or Case 2 upside down domain* V_{usd}*, then* $V \preceq_{Mac} V_{usd}$.

(3) *If* V *is an inductive commensurable domain with corresponding Case 3 upside down domain* V_{usd}*, then* $V_{usd} \preceq_{Mac} V$.

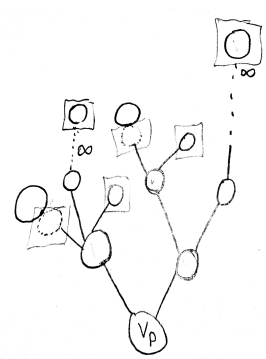
(4) *For every integrally closed domain closed domain between* $\mathbb{Z}_p[X]$ *and* $\mathbb{Q}[X]$*, every* p*-unitary valuation overring of* D *is comparable to a minimal element under* $\preceq_{Mac}$.

Notation 2.27 *Denote by* T_D *the set of valuation overrings of* D *which are minimal with respect to* $\preceq_{Mac}$. *Then, each* $V \in T_D$ *is not a Case 1 or Case 2 upside down domain.*

For reasons beyond the scope of this proposal, if we wish to build a minimal set of p-unitary valuation overrings of D with respect to $\preceq_{Mac}$ over a fixed commensurable valuation domain V, then we need only consider key polynomials corresponding to a finite set of maximal ideals of D_V when extending V.

2.5 Representations of integrally closed domains as locally finite trees

Using the ordering defined in the previous subsection, we're going to represent a given integrally closed domain D as a tree, and this will give us the desired "minimal" intersection. Of course, all the valuation overrings of D with $\preceq_{Mac}$ form a tree, but we'd like to reduce the size of this tree somehow. This is accomplished by constructing a tree $\mathcal{T}_D$ only including the elements of T_D (which must be Case 3 upside down, inductive, or limit) and the inductive valuations required to construct them. At the base of the tree will be the valuation v_p on $\mathbb{Q}$, branching from that will be the first stage inductive valuations that lead to the elements of T_D, and so on. The upside down domains don't quite fit as nicely in this construction; thus, we "fix" it by thinking of Case 3 upside down domains as "intermediate vertices" just below the commensurable domains they're constructed from but "infinitessimally close" to these corresponding commensurable domains. The elements of T_D will thus be the maximal vertices of this tree (cutting out the commensurable domains used to construct Case 3 upside down domains), and this tree will necessarily be locally finite. Indeed, if we take an arbitrary inductive commensurable valuation overring V and $V \in \mathcal{T}_D$, only finitely many $V_i \in T_D$ will be such that $V \preceq_{Mac} V_i$ since the whole collection T_D can only arise from a finite number of key polynomials extending V (this is a consequence of the topology on the space of valuation rings, and the specifics here are beyond the scope of this proposal). Below is a simple example of one such tree:



The ovals are the valuations used to construct the elements of T_D, which are boxed. Solid ovals above v_p at the bottom denote inductive valuations, unless appearing above a dotted line with ∞; these are limit domains. Dotted ovals that are just below solid ovals denote Case 3 upside down domains. Each segment denotes a stage of the inductive construction. Therefore, this integrally closed domain D represented by this tree is an intersection of two limit valuations, a stage 2 and a stage 3 inductive valuation, and two Case 3 upside down domains, arising from a stage 2 inductive domain and a stage 3 inductive domain, respectively.

Conversely, one might wonder if a converse construction might work. In other words, given a locally finite tree with vertices given by the kinds of valuation domains we've constructed, can we construct an integrally closed domain D that's an intersection of the maximal vertices? It turns out this is indeed true.

Let $\mathcal{T}$ be a locally finite tree with vertices ordered under $\preceq_{Mac}$ whose nonmaximal vertices are commensurable valuation domains and whose maximal vertices are either inductive, limit, or Case 3 valuation domains.

Lemma 2.28 [9, Lemma 3.1] *Let $V_1, ..., V_n \in T_p$ be a collection of valuation domains, each either inductive, limit, or Case 3 upside down. Set $D = V_1 \cap \cdots \cap V_n \cap \mathbb{Q}[X]$. Then, each p-unitary prime ideal of D contains exactly one of the centers of the V_i's on D.*

Theorem 2.29 [9, Theorem 3.2, Corollary 3.3] *Let $\mathcal{V} \subseteq T_p$ be a (not necessarily finite) collection of valuation domains, each either inductive, limit, or Case 3 upside down, that can be represented as the maximal vertices of a locally finite tree $\mathcal{T}$. Set $D := \cap_{V \in \mathcal{V}} D_V$.*

(a) *If P is a p-unitary prime ideal of D, then $D_P = (D_V)_Q$ for some $V \in \mathcal{V}$ and some p-unitary prime ideal Q of D_V. Also, there exists a p-unitary valuation overring (R, M) of D such that $D_P = (D_R)_{M \cap D_R}$.*

(b) *Let $W \in T_D$ and P be the center of W in D. Then, $D_P = (D_W)_Q$ for some p unitary prime ideal Q of D_W. Moreover, $D_W = D_P \cap \mathbb{Q}[X] = \cap_{P \in \Delta} D_P$, where $\Delta \subseteq \mathrm{Spec}(D)$ consists of P and all primes of D that are uppers to 0 (meaning they contract to 0 when contracted to $D \cap \mathbb{Q}[X]$. In particular, for an arbitrary integrally closed domain D between $\mathbb{Z}_p[X]$ and $\mathbb{Q}[X]$, since $D = \cap_{W \in T_D} D_W$, for any prime ideal P of D, there exists $W \in T_D$ and $Q \in \mathrm{Spec}(D_W)$ such that $D_P = (D_W)_Q$.*

(c) *Each $V \in \mathcal{V}$ is a p-unitary valuation overring of D minimal under $\preceq_{Mac}$ (i.e. $V \in T_D$).*

2.6 Classification results regarding integrally closed domains between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$

In this subsection, we use the results in the previous sections to establish criteria for various nice properties (e.g. Prüfer, Krull, Mori) to occur for a given integrally closed domain D between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ if we know its "minimal" representation as constructed using $\preceq_{Mac}$ and the locally finite tree.

Notation 2.30 *Given a domain D between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ and $p \in \mathbb{N}$ prime, the* **p-component** *D_p is the localization of D at the multiplicative set $\mathbb{Z} \setminus p\mathbb{Z}$. Moreover, $D = \cap_p D_p$ where p ranges over all positive prime integers and $\mathbb{Z}_p[X] \subseteq D_p$.*

It's a fact that D is integrally closed if and only if D_p is integrally closed for all prime integers p. Thus, the classification of the integrally closed polynomial overrings of $\mathbb{Z}_p[X]$ for every prime integer p gives us significant insight in the classification of all integrally closed domains between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$.

Among the first results obtained was the following, due to Abhyankar, Heinzer, and Eakin.

Theorem 2.31 [1, Theorem 5.7] *Let $p \in \mathbb{Z}$ be a fixed prime integer. Let $V_1, V_2, ..., V_n$ be DVR overrings of $\mathbb{Z}_p[X]$ such that $V_i \cap \mathbb{Q} = \mathbb{Z}_p$ for each i. Let $D = V_1 \cap V_2 \cap \cdots \cap V_n \cap \mathbb{Q}[X]$. Then, D is a Dedekind domain provided that the residue field of each V_i is algebraic over $\mathbb{F}_p$. If one of the residue fields is not algebraic over $\mathbb{F}_p$, then D is a two-dimensional Noetherian domain (in particular, D cannot be Prüfer).*

Recall that T_p is the set of all valuation overrings of $\mathbb{Z}[X]$ containing p as a nonunit. Given, $V \in T_p$, we define $D_V := V \cap \mathbb{Q}[X]$.

Proposition 2.32 [9, Proposition 4.1] *Given $V \in T_p$, D_V is a Prüfer domain if and only if V is a limit valuation domain.*

Theorem 2.33 [9, Theorem 4.2, Corollary 4.4] *Given a domain D such that $\mathbb{Z}_p[X] \subseteq D \subseteq \mathbb{Q}[X]$, D is Prüfer if and only if D is integrally closed and all the p-unitary overrings of D are limit. If $\mathbb{Z}[X] \subseteq D \subseteq \mathbb{Q}[X]$, D is Prüfer if and only if D is integrally closed and all $V \in T_p$ that are overrings of D are limit for each prime integer p.*

Theorem 2.34 [9, Theorem 4.5] *Let $\mathcal{T}$ be a locally finite tree of inductive domains in T_p such that every branch of $\mathcal{T}$ converges to a limit valuation domain. Let $\mathcal{W} = \{W_i\}_{i \in I}$ be the collection of these limit valuation domains. Then, $D := \cap_{i \in I} W_i \cap \mathbb{Q}[X]$ is Prüfer.*

Theorem 2.35 [9, Theorem 5.1] *Let D be integrally closed, and suppose $\mathbb{Z}_p[X] \subseteq D \subseteq \mathbb{Q}[X]$. Then D is Krull if and only if $D = V_1 \cap \cdots \cap V_n \cap \mathbb{Q}[X]$, where $V_i \in T_p$ is inductive commensurable or finite limit. In particular, in this case, D is Noetherian, and D is the intersection of a semilocal PID with $\mathbb{Q}[X]$.*

Definition 2.36 *If I is an ideal in a domain D such that $I = (D : (D : I))$, then we say I is a* **divisorial ideal.** *A* **Mori domain** *is a domain that satisfies the ascending chain condition on divisorial ideals.*

Proposition 2.37 [9, Proposition 5.5] *Given $V \in T_p$, D_V is Mori if and only if V is inductive commensurable, upside-down, or finite limit.*

Theorem 2.38 [9, Theorem 5.6, Corollary 5.7] *Let D be an integrally closed domain such that $\mathbb{Z}_p[X] \subseteq D \subseteq \mathbb{Q}[X]$. Then, D is Mori if and only if $D = V_1 \cap \cdots \cap V_n \cap \mathbb{Q}[X]$, where each $V_i \in T_p$ is either inductive commensurable, upside down, or finite limit. If $\mathbb{Z}[X] \subseteq D \subseteq \mathbb{Q}[X]$ and D is Mori, then each p-component D_p has the structure given in the previous sentence.*

Definition 2.39 *If D is a domain that can be written as $D = \cap_{P \in \mathcal{P}} D_P$, where $\mathcal{P} \subseteq D$, each D_P is a valuation domain, and the intersection is locally finite (meaning every element of the quotient field K of D is a non-unit in at most finitely many of the D_P), then we say D is a* **Krull-type domain.**

Proposition 2.40 [9, Corollary 5.9] *Let D be integrally closed, and suppose $\mathbb{Z}_p[X] \subseteq D \subseteq \mathbb{Q}[X]$. Then, D is Krull-type if and only if $D = V_1 \cap \cdots \cap V_n \cap \mathbb{Q}[X]$, where $V_i \in T_p$ is inductive commensurable or limit.*

If $\mathbb{Z}[X] \subseteq D \subseteq \mathbb{Q}[X]$ and D is of Krull-type, then each D_p is. The converse of this statement is not true, as Example 5.3 of [9] shows.

3 The case of more than one variable

Given that the previous sections established very nice results about the integrally closed domains between $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ by constructing and classifying valuations on $\mathbb{Q}[X]$, we hope to establish similarly nice results (though perhaps much more modest) in polynomial rings in more variables. We begin this section by noting what fails in the higher dimensional cases and then proceed by examining a new class of valuation rings, the $V_{p,(\alpha,\beta)}$ to be defined in the subsequent subsection. Then, we'll conclude the section by stating an important result by Hiroshi Inoue in [6] having to do with the construction of discrete valuations on $K[X,Y]$ for an arbitrary field K and explore potential consequences of the result.

3.1 What fails in the higher dimensional cases

To begin, note that $\mathbb{Q}[X,Y]$ is not a Prüfer domain (since the localization $\mathbb{Q}[X,Y]_{(X,Y)}$ is not a valuation domain, as one sees by comparing the prime ideals (X) and (Y) inside it), and therefore, any subring of $\mathbb{Q}[X,Y]$ with the same quotient field (namely, any integrally closed domain D such that $\mathbb{Z}_p[X,Y] \subseteq D \subseteq \mathbb{Q}[X,Y]$) is not Prüfer. This is due to the fact that any overring of a Prüfer domain is Prüfer.

Second, for any $n \in \mathbb{N}$, $\mathbb{Z}_p[X_1, ..., X_n]$ is Noetherian and hence Jaffard, meaning it has valuation overrings of all dimensions $\leq n+1$. Because $\mathbb{Z}_p[X]$ could only have valuation rings of dimension 1 or 2, this greatly restricted behavior and enabled us to give nicer classification theorems. In our analysis of the single variable case, we saw that any integrally closed domain D such that $\mathbb{Z}_p[X] \subseteq D \subseteq \mathbb{Q}[X]$ was locally a finite stage inductive (commensurable or incommensurable) domain, a finite limit domain, or a Case 3 upside down domain. Adding more variables significantly complicates matters.

Additionally, it's apparent that $\preceq_{Mac}$ extends naturally to the case of multiple variables, and due to the patch topology on valuation overrings and Zorn's lemma, minimal elements exist with respect to $\preceq_{Mac}$ in the case of multiple variables. However, the tree structure is much less apparent. In the single variable case, for V at the top of the tree, the residue field of V was algebraic over a finite field and D_V was Prüfer, while for V at the the bottom, the residue fields are smaller and D_V was Noetherian. By introducing more variables, the residue field structure at the top of the tree is not clear. For instance, the question of when a valuation domain which is maximal with respect to $\preceq_{Mac}$ has transcendence degree 1 or 2 over a finite field is not readily apparent.

Moreover, given a valuation v on $\mathbb{Q}(X_1, ..., X_n)$, restricting v to $\mathbb{Q}(X_i)$ gets us one of the types of valuations described in the previous sections, but the independence of the variables seems to preclude any sort of "simple characterization" analogous to the one-variable case. For instance, if $v|_{\mathbb{Q}(X_1)}$ is Case 1 upside down, $v|_{\mathbb{Q}(X_2)}$ is Case 3 upside down, $v|_{\mathbb{Q}(X_3)}$ is finite stage inductive commensurable, $v|_{\mathbb{Q}(X_4)}$ is infinite limit, and $v|_{\mathbb{Q}(X_5)}$ is finite limit, for instance, what can we say about the structure of the valuation ring corresponding to v? If another valuation w were constructed in a completely different way, how could we compare v and w under $\preceq_{Mac}$? These questions are unresolved.

Furthermore, constructing upside down domains in $\mathbb{Q}(X)$, we started with a discrete valuation and exploited the simple structure of $F(Y)$ (where F is a finite extension of $\mathbb{F}_p$) and its valuation rings. However, in constructing 3-dimensional valuation rings in $\mathbb{Q}(X, Y)$, for instance, how an extension of this procedure would work and its applicability are unknown, and if one uses pullbacks, the residue fields used are more complicated than the aforementioned $F(Y)$. To begin the study of this greater variety of valuation rings, then, we will focus our attention on specific "exotic" examples of valuation rings V containing $\mathbb{Z}_p[X, Y]$ and their resulting polynomial domains $D_V = V \cap \mathbb{Q}[X, Y]$ in the next subsection without regard to whether they can be constructed via inductive valuations like in the 1 variable case.

3.2 $V_{p,(\alpha,\beta)}$ and $D_{p,(\alpha,\beta)}$

In this subsection, we use a construction from [8]. To motivate this, we note that in the single variable case one class of valuation ring that arises naturally from the study of integer-valued polynomials is the collection of $V_{p,\alpha} := \{\varphi \in \mathbb{Q}(X) \ : \ \varphi(\alpha) \in \hat{\mathbb{Z}}_p\}$, where p is a fixed prime and $\alpha := \sum_{i\geq 0} a_i p^i$ ($0 \leq a_i \leq p-1$ for all i) is an element of $\hat{\mathbb{Z}}_p$. As shown on pages 146 and 147 of [9], $V_{p,\alpha}$ is an infinite limit valuation domain constructed by the sequence of key polynomials $\phi_k(X) = X - \left(\sum_{i<k} a_i p^i\right)$ with values $v_k(\phi_k) = k$. Moreover, its residue field is $\mathbb{F}_p$. In several variables, an analogue to this ring can be defined and studied, and it has considerably more nuance which this section will explore.

Definition 3.1 *Let p be a prime number and $\alpha, \beta \in \hat{\mathbb{Q}}_p$. Define the valuation ring $V_{p,(\alpha,\beta)}$ to be*

$$V_{p,(\alpha,\beta)} := \{\, \phi(X, Y) \in \mathbb{Q}(X, Y) : v_p(\phi(\alpha, \beta)) \geq 0 \,\}.$$

First, some basic facts about these rings.

Proposition 3.2 [8, Proposition 2.9] *Let p, α, β be as in the above definition. Then, $V_{p,(\alpha,\beta)}$ has maximal ideal generated by p and residue field $\mathbb{F}_p$. If α and β are transcendental over $\mathbb{Q}$ and algebraically independent, then $V_{p,(\alpha,\beta)}$ is one dimensional.*

Notation 3.3 *Let $a, b \in \mathbb{Z}$. We denote $V_{0,(a,b)} = \{\phi(X, Y) \in \mathbb{Q}(X, Y) \mid \phi(a, b) \in \mathbb{Q}\}$ and $V_{0,(a,y)} = \{\phi \in \mathbb{Q}(X, Y) \mid \phi(a, y) \in \mathbb{Q}(Y)\}$. Note that for any prime p prime and $a, b \in \mathbb{Z}$ that $V_{0,(a,y)} \supseteq V_{0,(a,b)} \supseteq V_{p,(a,b)}$.*

The principal object of interest in this section is $D_{p,(\alpha,\beta)} := V_{p,(\alpha,\beta)} \cap \mathbb{Q}[X, Y]$. If $\alpha, \beta \in \hat{\mathbb{Z}}_p$ (the p-adic integers), then $\mathbb{Z}_p[X, Y] \subseteq D_{p,(\alpha,\beta)}$. Moreover, since for each $\alpha \in \hat{\mathbb{Q}}_p$, there exists a unique smallest integer $n \in \mathbb{N}$ such that $p^n\alpha \in \hat{\mathbb{Z}}_p$, which gives us an isomorphism between $D_{p,(\alpha,\beta)}$ and $D_{p,(p^n\alpha,\beta)}$ via $X \mapsto X/p^n$. Therefore, it suffices to consider $\alpha, \beta \in \hat{\mathbb{Z}}_p$, which gives us our desired hypothesis $\mathbb{Z}_p[X, Y] \subseteq D_{p,(\alpha,\beta)} \subseteq \mathbb{Q}[X, Y]$. We attempt to give the nicest possible decompositions of these rings with various choices of α and β.

Case 1: $\alpha = \beta = 0$. Then, $D_{p,(0,0)} = \mathbb{Z}_p + (X, Y)\mathbb{Q}[X, Y]$.
Case 2: $\alpha = 0$, $\beta = 1$. We may decompose this as $D_{p,(0,1)} = X\mathbb{Q}[X, Y] + D_1$, where

$$D_1 = \left\{ f = \sum_{i=0}^{n} a_i Y^i \in \mathbb{Q}[Y] \ : \ \sum_{i=0}^{n} a_i \in \mathbb{Z}_p \right\}.$$

Moreover, for any $f(Y) = \sum_{i=1}^{n} a_i Y^i \in D_1$, if $a_i = c_i p^{-n_i}$ where $c_i \in \mathbb{Z}_p$ and $n_i \in \mathbb{N} \cup \{0\}$, then

$$\left(\sum_{i=1}^{n} c_i \frac{1 - Y^i}{p^{n_i}} \right) + f(Y) = \sum_{i=0}^{n} a_i \in \mathbb{Z}_p,$$

and therefore f is in the $\mathbb{Z}_p$-module generated by 1 and $\left\{\frac{Y^i-1}{p^n}\right\}_{i,n\in\mathbb{N}}$. Since this $\mathbb{Z}_p$-module is clearly a subset of D_1, we therefore have that D_1 is equal to this $\mathbb{Z}_p$-module.
Case 3: $\alpha = 0$, $\beta = \frac{c}{d} \in \mathbb{Q}\setminus\{0\}$. We may decompose this as $D_{p,(0,\beta)} = X\mathbb{Q}[X,Y] + D_\beta$, where

$$D_\beta = \left\{ f = \sum_{i=0}^{n} a_i \left(\frac{d}{c}Y\right)^i \in \mathbb{Q}[Y] \,:\, \sum_{i=0}^{n} a_i \in \mathbb{Z}_p \right\}.$$

Moreover, D_β is generated by the set

$$\left\{ \frac{\left(\frac{d}{c}Y\right)^i - 1}{p^n} \right\}_{i,n\in\mathbb{N}}$$

as a $\mathbb{Z}_p$-module by the same reasoning as in Case 2.
Case 4: $\alpha = \beta = 1$. In this case, $D_{p,(1,1)} = \mathrm{Int}(\{(1,1)\}, \mathbb{Z}_p)$ and the set

$$\left\{ 1, \frac{X^kY^l - 1}{p^n} \right\}_{k,l,n\in\mathbb{N}}$$

generates $D_{p,(1,1)}$ as a $\mathbb{Z}_p$-module. Moreover, it is immediate that $D_{p,(1,1)}$ is the set of polynomials $f(X,Y) = \sum_{i,j=0}^{n} a_{ij}X^iY^j \in \mathbb{Q}[X,Y]$ ($n = \max\{\deg_X(f), \deg_Y(f)\}$) such that $\sum a_{ij} \in \mathbb{Z}_p$.

Proof. The following proof, an improvement over my original proof, is due to K.A. Loper. Let

$$f(X,Y) = \sum_{i,j=0}^{n} a_{ij}X^iY^j,$$

where $n = \max\{\deg_X(f), \deg_Y(f)\}$, be an element of $D_{p,(1,1)}$. We may write each a_{ij} as $a_{ij} = c_{ij}p^{-n_{ij}}$, where $c_{ij} \in \mathbb{Z}_p$ and $n_{ij} \in \mathbb{N}\cup\{0\}$. Then, since each term $c_{kl}\frac{1-X^kY^l}{p^{n_{kl}}}$ is an element of $D_{p,(1,1)}$, we therefore know

$$\sum_{k,l=1}^{n} \left(c_{kl} \cdot \frac{1 - X^kY^l}{p^{n_{kl}}} \right) + c_{10} \cdot \frac{1-X}{p^{n_{10}}} + c_{01} \cdot \frac{1-Y}{p^{n_{01}}} + f(X,Y) \in D_{p,(1,1)}.$$

However, we know that this is equal to the sum of the coefficients of f (the a_{ij}), $f(1,1)$, which is in $\mathbb{Z}_p$, so by moving all but f in the above expression to the "right hand side," we see that

$$f(X,Y) = \sum_{k,l=1}^{n} \left(c_{kl} \cdot \frac{X^kY^l - 1}{p^{n_{kl}}} \right) + c_{10} \cdot \frac{X-1}{p^{n_{10}}} + c_{01} \cdot \frac{Y-1}{p^{n_{01}}} + \sum_{i,j=0}^{n} a_{ij},$$

which means that $f(X,Y)$ is in the given $\mathbb{Z}_p$-module, as desired.

■

Case 5: $\alpha = \frac{a}{b}, \beta = \frac{c}{d} \in \mathbb{Q}\setminus\{0\}$. In this case, just like in Case 4,

$$\left\{ 1, \frac{(\frac{b}{a}X)^k(\frac{d}{c}Y)^l - 1}{p^n} \right\}_{k,l,n\in\mathbb{N}}$$

generates $D_{p,(\alpha,\beta)}$ as a $\mathbb{Z}_p$-module, and we show this by adapting the proof provided with the alternate decomposition

$$f(X,Y) = \sum_{i,j=0}^{n} e_{ij} \left(\frac{b}{a}X\right)^i \left(\frac{d}{c}Y\right)^j.$$

For all remaining cases, if $\alpha \notin \mathbb{Q}$ and/or $\beta \notin \mathbb{Q}$, then we set $\alpha = a_{-m}p^{-m} + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots$ and $\beta = b_{-n}p^{-n} + \cdots + b_0 + b_1 p + b_2 p^2 + \cdots$. All remaining cases are purely conjecture. The right hand sides may very well be strictly smaller subrings.

Case 6: $\alpha = \frac{a}{b} \in \mathbb{Q} \setminus \{0\}$, $\beta \notin \mathbb{Q}$, β algebraic with minimal polynomial $g(Y)$. In this case,

$$D_{p,(\alpha,\beta)} = (bX - a,\, g(Y))\mathbb{Q}[X,Y] +$$
$$\mathbb{Z}_p\left[\frac{b}{a}X, \left\{p^m f(Y), \frac{f(Y) - \sum_{i=-m}^{r} c_i p^i}{p^{r+1}} \;:\; f(Y) \in \mathbb{Q}[Y],\ f(\beta) = \sum_{i=-m}^{\infty} c_i p^i,\ 0 \le c_i \le p-1\ \forall i,\ r \ge -m\right\}\right].$$

Case 7: $\alpha = 0$, $\beta \notin \mathbb{Q}$, β algebraic with minimal polynomial $g(Y)$. In this case,

$$D_{p,(\alpha,\beta)} = (X, g(Y))\mathbb{Q}[X,Y] +$$
$$\mathbb{Z}_p\left[\left\{p^m f(Y), \frac{f(Y) - \sum_{i=-m}^{r} c_i p^i}{p^{r+1}} \;:\; f(Y) \in \mathbb{Q}[Y],\ f(\beta) = \sum_{i=-m}^{\infty} c_i p^i,\ 0 \le c_i \le p-1\ \forall i,\ r \ge -m\right\}\right].$$

Case 8: $\alpha, \beta \notin \mathbb{Q}$ but are algebraic with minimal polynomials $g(X)$ and $h(Y)$ over $\mathbb{Q}$. If $\mathbb{Q}[\alpha, \bar{\alpha}] \cap \mathbb{Q}[\beta, \bar{\beta}] = \mathbb{Q}$, then

$$D_{p,(\alpha,\beta)} = (g(X),\, h(Y))\mathbb{Q}[X,Y] +$$
$$\mathbb{Z}_p\left[\left\{p^m f(X,Y), \frac{f(X,Y) - \sum_{i=-m}^{r} c_i p^i}{p^{r+1}} \;:\; f(X,Y) \in \mathbb{Q}[X,Y],\ f(\alpha,\beta) = \sum_{i=-m}^{\infty} c_i p^i,\ 0 \le c_i \le p-1\ \forall i,\ r \ge -m\right\}\right].$$

Case 9: $\alpha = \frac{a}{b} \in \mathbb{Q}$, β transcendental. In this case,

$$D_{p,(\alpha,\beta)} = (bX - a)\mathbb{Q}[X,Y] +$$
$$\mathbb{Z}_p\left[\frac{b}{a}X, \left\{p^m f(Y), \frac{f(Y) - \sum_{i=-m}^{r} c_i p^i}{p^{r+1}} \;:\; f(Y) \in \mathbb{Q}[Y],\ f(\beta) = \sum_{i=-m}^{\infty} c_i p^i,\ 0 \le c_i \le p-1\ \forall i,\ r \ge -m\right\}\right].$$

Case 10: $\alpha \notin \mathbb{Q}$ but algebraic with minimal polynomial $g(X)$ and β transcendental. In this case,

$$D_{p,(\alpha,\beta)} = g(X)\mathbb{Q}[X,Y] +$$
$$\mathbb{Z}_p\left[\left\{p^m f(X,Y), \frac{f(X,Y) - \sum_{i=-m}^{r} c_i p^i}{p^{r+1}} \;:\; f(X,Y) \in \mathbb{Q}[X,Y],\ f(\alpha,\beta) = \sum_{i=-m}^{\infty} c_i p^i,\ 0 \le c_i \le p-1\ \forall i,\ r \ge -m\right\}\right].$$

Case 11: α and β transcendental and algebraically independent. In this case,

$$D_{p,(\alpha,\beta)} =$$
$$\mathbb{Z}_p\left[\left\{p^m f(X,Y), \frac{f(X,Y) - \sum_{i=-m}^{r} c_i p^i}{p^{r+1}} \;:\; f(X,Y) \in \mathbb{Q}[X,Y],\ f(\alpha,\beta) = \sum_{i=-m}^{\infty} c_i p^i,\ 0 \le c_i \le p-1\ \forall i,\ r \ge -m\right\}\right].$$

Case 12: α and β transcendental and algebraically dependent over $\mathbb{Q}$. In this case, if our dependence relation is $g(\alpha,\beta) = 0$ for some $g(X,Y) \in \mathbb{Q}[X,Y]$, then

$$D_{p,(\alpha,\beta)} = g(X,Y)\mathbb{Q}[X,Y] +$$
$$\mathbb{Z}_p\left[\left\{p^m f(X,Y), \frac{f(X,Y) - \sum_{i=-m}^{r} c_i p^i}{p^{r+1}} \;:\; f(X,Y) \in \mathbb{Q}[X,Y],\ f(\alpha,\beta) = \sum_{i=-m}^{\infty} c_i p^i,\ 0 \le c_i \le p-1\ \forall i,\ r \ge -m\right\}\right].$$

Question 3.4 *Can these fractions in Cases 6 to 12 involving arbitrary polynomials with rational coefficients be found in "smaller" (meaning simpler to constuct) rings like*

$$\mathbb{Z}_p\left[p^mX, p^nY, \left\{\frac{X-\sum_{i=-m}^{r} a_ip^i}{p^{r+1}}\right\}_{r\geq -m}, \left\{\frac{Y-\sum_{i=-n}^{s} b_ip^i}{p^{s+1}}\right\}_{s\geq -n}\right]$$

or

$$\mathbb{Z}_p\left[\left\{p^mX^kY^l, \frac{X^kY^l-\sum_{i=-m}^{r} c_ip^i}{p^{r+1}} \;:\; \alpha^k\beta^l=\sum_{i=-m}^{\infty} c_ip^i,\ 0\leq c_i\leq p-1\ \forall i,\ r\geq -m,\ k,l\in\mathbb{N}\right\}\right]?$$

This seems unlikely.

Question 3.5 *As we saw at the start of the section, we were able to construct $V_{p,\alpha}$ in one variable as a finite stage inductive valuation or as a limit valuation. In any of these cases just discussed, would we be able to construct $V_{p,(\alpha,\beta)}$ similarly? On a related note, since $V_{p,(\alpha,\beta)}$ is a "join" of $V_{p,\alpha}$ in the X variable and $V_{p,\beta}$ in the Y variable, the structure of these should determine the structure of $V_{p,(\alpha,\beta)}$, but exactly how is not entirely clear. If $V_{p,\alpha}$ is inductive incommensurable and $V_{p,\beta}$ is finite limit, for instance, what could be said about the structure of $V_{p,(\alpha,\beta)}$? To answer these questions, perhaps Inoue's Principal Problem (to be discussed later in this proposal) may be of use.*

3.3 $D_{p,(\alpha,\beta)}$ through the lens of integer-valued polynomials

In the above enumeration of cases, we were considering the module structure of these rings $D_{p,(\alpha,\beta)}$. We will now attempt to understand their ideal structure. First, we introduce some new notation, and then we will make some observations that follow from the fact that these rings can be obtained from rings of integer-valued polynomials.

Notation 3.6 *Let D be a domain with quotient field K, $n\in\mathbb{N}$, and $\underline{E}\subseteq K^n$. Then, we define*

$$\mathrm{Int}(\underline{E},D)=\{f\in K[X_1,...,X_n]\mid f(\underline{E})\subseteq D\}.$$

Observe that $D_{p,(\alpha,\beta)}=\mathrm{Int}(\{(\alpha,\beta)\},\hat{\mathbb{Z}}_p)\cap\mathbb{Q}[X,Y]$. Therefore, using the better known ideal structure of these rings of integer-valued polynomials, we may obtain some insight about the ideal structure of $D_{p,(\alpha,\beta)}$. We will now establish some of those results about integer-valued polynomials.

Theorem 3.7 [3, Theorem XI.2.9] *Let $(D,\mathfrak{m})$ be a one-dimensional local Noetherian domain with finite residue field that is integrally closed. If D is analytically irreducible (i.e. its $\mathfrak{m}$-adic completion is a domain), then the prime ideals of $\mathrm{Int}(\underline{E},D)$ above $\mathfrak{m}$ are in one-to-one correspondence with the elements of the $\mathfrak{m}$-adic completion $\hat{\underline{E}}$ of $\underline{E}$: to each element $\underline{\alpha}\in\hat{\underline{E}}$ corresponds the maximal ideal*

$$\mathfrak{M}_{\mathfrak{m},\underline{\alpha}}:=\{f\in\mathrm{Int}(\underline{E},D)\mid f(\underline{\alpha})\in\hat{\mathfrak{m}}\}.$$

Proposition 3.8 [3, Proposition XI.2.10] *Assume all hypotheses and notation from the previous theorem.*

(i) *If $\mathfrak{P}$ is a prime ideal of $K[X_1,...,X_n]$, then $\mathfrak{P}\cap\mathrm{Int}(\underline{E},D)\subseteq\mathfrak{M}_{\mathfrak{m},\underline{\alpha}}$ if and only if $\underline{\alpha}$ is a common root of the polynomials contained in $\mathfrak{P}$. In fact, this occurs if and only if*

$$\mathfrak{P}\cap\mathrm{Int}(\underline{E},D)\subseteq\mathfrak{O}_{\underline{\alpha}}:=\{f\in\mathrm{Int}(\underline{E},D)\mid f(\underline{\alpha})=0\}\subseteq\mathfrak{M}_{\mathfrak{m},\underline{\alpha}}.$$

(ii) *The ideal $\mathfrak{M}_{\mathfrak{m},\underline{\alpha}}$ is of height $n+1-trdeg_K K(\underline{\alpha})$ and $\mathfrak{O}_{\underline{\alpha}}$ is of height $n-trdeg_K K(\underline{\alpha})$.*

We will be applying these two results to the case where $D=\hat{\mathbb{Z}}_p$ and $\underline{E}=\{(\alpha,\beta)\}\subseteq\hat{\mathbb{Z}}_p\times\hat{\mathbb{Z}}_p$.

Corollary 3.9 (i) *$B_{p,(\alpha,\beta)}=\{f\in D_{p,(\alpha,\beta)}\mid f(\alpha,\beta)\in p\hat{\mathbb{Z}}_p\}$ is a prime ideal of $D_{p,(\alpha,\beta)}$ lying over $p\hat{\mathbb{Z}}_p$*

(ii) *$O_{p,(\alpha,\beta)}=\mathfrak{O}_{(\alpha,\beta)}\cap\mathbb{Q}[X,Y]\subseteq D_{p,(\alpha,\beta)}$ is an upper to 0 and is of either height one or two (because $\mathfrak{O}_{\underline{\alpha}}$ is of height two since $\alpha,\beta\in\hat{\mathbb{Z}}_p$).*

Question 3.10 *When exactly is the ideal $O_{p,(\alpha,\beta)}$ of height one, and when is it height two? If it's height two, what are all the height one ideals below it?*

If α or β is transcendental, then $O_{p,(\alpha,\beta)}$ is of height one. Since $\hat{\mathbb{Q}}_p[X,Y]$ is a unique factorization domain, its height one prime ideals are principal, so we obtain height one prime ideals of $D_{p,(\alpha,\beta)}$ by contracting these prime ideals.

3.4 Inoue finding and its utility

In this subsection, we give an important result by Inoue on discrete valuations of $K[X,Y]$ and attempt to derive some consequences.

Theorem 3.11 [6, Principal Problem] *Let K be a field with discrete valuation v_{00}, and let $v_{p0} = [v_{00}, v_{10}(X) = \mu_1, v_{20}(\phi_2(X)) = \mu_2, ..., v_{p0}(\phi_p(X)) = \mu_p]$ and $v_{0q} = [v_00, v_{01}(Y) = \nu_1, v_{02}(\psi_2(Y)) = \nu_2, ..., v_{0q}(\psi_q(Y)) = \nu_q]$ be inductive commensurable valuations extended from v_{00} on $K[X]$ and $K[Y]$, respectively. Then, there is a valuation w on $K[X,Y]$ extending v_{p0} and v_{0q}.*

Inoue's proof is long and opaque, so what exactly "w" is may be missed. We aim for an alternative construction that is more basic and insightful.

Construction 3.12 *I believe that one such desired extension w to $K(X,Y)$ can be defined (inductively) by*

$$w(f) := \inf_{\alpha \in A} \left\{ w\left(\sum_{i>0,j>0} c_{ij}\phi_p^i \psi_q^j + r_{pq}(X,Y) \right) \right\}$$

where A is the set of all decompositions

$$f(X,Y) = \sum_{i>0,j>0} c_{ij}\phi_p^i \psi_q^j + r_{pq}(X,Y)$$

where $\deg_X(c_{ij}) < \deg_X(\phi_p)$ and $\deg_Y(c_{ij}) < \deg_Y(\psi_q)$ for all i and all j, $\deg_X(r_{pq}) < \deg_X(\phi_p)$, and $\deg_Y(r_{pq}) < \deg_Y(\psi_q)$, and where we define

$$w\left[\sum_{i>0,j>0} c_{ij}\phi_p^i \psi_q^j + r_{pq}(X,Y) \right] := \min\{\ \{w(c_{ij}) + i\mu_p + j\nu_q\}_{i,j>0}, \{w(r_{pq}(X,Y))\}\}$$

and where we decompose

$$c_{ij} = \sum_{m>0,n>0} d_{mn,ij}\phi_{p-1}^m \psi_{q-1}^n + r_{p-1,q-1,ij}$$

$$r_{pq} = \sum_{m>0,n>0} e_{mn}\phi_{p-1}^m \psi_{q-1}^n + r_{p-1,q-1}$$

and iterate this process similarly.

The reason that this construction is so complicated is that the lack of uniqueness present in decompositions obtained from the division algorithm for $K[X,Y]$.

Example 3.13 *Dividing $x^2y + xy^2 + y^2$ by $xy - 1$ and $y^2 - 1$ we obtain two decompositions*

$$x^2y + xy^2 + y^2 = (x+1)(y^2-1) + x(xy-1) + [2x+1]$$

and

$$x^2y + xy^2 + y^2 = (x+y)(xy-1) + 1 \cdot (y^2-1) + [x+y+1].$$

Because of this, attempting to prove that $w(fg) = w(f) + w(g)$ has been very difficult.

Assuming that this "solution" to Inoue's "Principal Problem" is valid, we may be able to extend it, to limit valuations of inductive commensurable sequences.

Question 3.14 *Let $v_{00} = v_p$. Suppose $w = \lim_{p\to\infty} w_p$, where $w_p = v_{p0} = [v_{00}, v_{10}(X) = \mu_1, v_{20}(\phi_2(X)) = \mu_2, ..., v_{p0}(\phi_p(X)) = \mu_p]$ is a pth stage inductive commensurable domain and suppose that $u = \lim_{q\to\infty} u_q$, where $u_q = v_{0q} = [v_{00}, v_{01}(Y) = \nu_1, v_{02}(\psi_2(Y)) = \nu_2, ..., v_{0q}(\psi_q(Y)) = \nu_q]$ is a qth stage inductive commensurable domain. For each pair $(p, q) \in \mathbb{N} \times \mathbb{N}$, we can apply the Principal Problem to w_p and u_q to obtain a discrete valuation $\hat{v}_{pq}$ on $\mathbb{Q}[X, Y]$. If we choose $m \leq p$ and $n \leq q$, does $\hat{v}_{pq}$ extend $\hat{v}_{mn}$? If so, for a fixed increasing sequence $\mathcal{S} = \{(p_i, q_i)\}_{i\in\mathbb{N}} \subseteq \mathbb{N} \times \mathbb{N}$ tending to infinity, can we define a limit valuation $\hat{v}_{\mathcal{S}} = \lim_{i\to\infty} \hat{v}_{p_i q_i}$? If $\mathcal{S}$ and $\mathcal{T}$ are two such sequences, is it the case $\hat{v}_{\mathcal{S}} = \hat{v}_{\mathcal{T}}$?*

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