1 Second Order Ordinary Differential Equations

Definition 1.1 *An* **ordinary differential equation** *(ODE) is an equation with derivatives of a given function $y(t)$ of a parameter t. An ODE is* **second order** *if y, y', and y'' appear in the equation but not higher order derivatives. A second order ODE is* **linear** *if y, y', and y'' only appear to the first power.*

The general form of a second order linear ODE is

$$y''(t) + p(t)y't(t) + g(t)y(t) = f(t) \tag{1}$$

where $p(t)$, $g(t)$, and $f(t)$ are continuous on an interval I. If $f(t)$ is identically 0, then we say that the ODE is **homogeneous**. If not, then the equation is **nonhomogeneous**.

Facts:

1. If (1) is homogeneous and y_1 and y_2 are two linearly independent solutions to (1), then ALL solutions to (1) are of the form $c_1y_1 + c_2y_2$ (where c_1 and c_2 are arbitrary real numbers).

2. The following pairs of functions are linearly independent: (a) $\{\sin(at), \cos(bt)\}$ for all nonzero a and b; (b) $\{e^{at}, e^{bt}\}$ for $a \neq b$; (c) $\{t^p, t^q\}$ if $p \neq q$.

3. If (1) is nonhomogeneous and y_p is a solution to (1) as written (we call this y_p a **particular solution** to (1)), then all solutions to (1) are of the form $y_p + y_h$, where y_h is a solution to (1)'s homogeneous counterpart (the form of this solution is given in Fact 1.).

4. Given (1) holding on an interval I containing 0 and given constants A and B where $y(0) = A$ and $y'(0) = B$, we can find a UNIQUE solution to (1), as this information enables us to solve for the c_1 and c_2 that we treated as arbitrary in Fact 1.

2 Complex Numbers

Definition 2.1 *A* **complex number** *is a number z of the form $z = x + iy$ where $i = \sqrt{-1}$ and x and y are real numbers. Geometrically, we think of $x+iy$ as the pair $\langle x, y\rangle$ in $\mathbb{R}^2$. We call x the* **real part** *of z, denoted Re(z), and y the* **imaginary part** *of z, denoted Im(z). The number $\bar{z} = x - iy$ is called the* **complex conjugate** *of z. Treated as a vector in $\mathbb{R}^2$, $x + iy$ is completely determined by its* **modulus** *(length) $|z| = \sqrt{x^2 + y^2} = \sqrt{z \cdot \bar{z}}$ and* **argument** *θ, which is the counterclockwise angle the vector $\langle x, y\rangle$ forms with the positive x-axis. Notice that $z = |z|(\cos(\theta) + i\sin(\theta))$. Then, because of* **Euler's formula** *$e^{i\theta} = \cos(\theta) + i\sin(\theta)$ (this is proven by taking the Maclaurin series for $e^{i\theta}$ and separating real and imaginary parts), we have $z = |z|e^{i\theta}$. This form of a complex number is generally easier to do multiplication and division with if the arguments are known angles. Otherwise, multiplication is done by FOILing, and a quotient z/w is found by taking $\frac{z}{w} = \frac{z}{w} \cdot \frac{\bar{w}}{\bar{w}} = \frac{z \cdot \bar{w}}{|w|^2}$.*

We can now state a very simple but beautiful fact:

$$e^{i\pi} + 1 = 0.$$

Fact: Notice that $e^{i\theta} = e^{i(\theta+2k\pi)}$ for any integer k

Theorem 2.2 De Moivre's Theorem *If $z = re^{i\theta} = re^{i(\theta+2k\pi)}$ (for any integer k, as we stated above), then there are n "n^{th} roots" of z:*

$$z^{1/n} = r^{1/n}e^{\frac{i(\theta+2k\pi)}{n}} = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{theta + 2k\pi}{n}\right)\right],$$

for $k = 0, 1, ..., n - 1$.