1 Algebraic Properties of Matrix Operations

Definition 1.1 *If $A = (a_{ij})$ is an $(m \times n)$ matrix, then the* **transpose** *of A, denoted A^T is the $(n \times m)$ matrix $A^T = (b_{ij})$, where $b_{ij} = a_{ji}$ for all i and j, $1 \leq i \leq m$ and $1 \leq j \leq n$. A matrix A is* **symmetric** *if $A = A^T$.*

Theorem 1.2 [Theorems 8, 9, 10 p.360-362] *1. If A and B are $(m \times n)$ matrices and C is an $(n \times p)$ matrix, then $(A+B)C = AC + BC$.*
2. If A is an $(m \times n)$ matrix and B and C are $(n \times p)$ matrices, then $A(B+C) = AB + AC$.
3. If r and s are scalars and A is an $m \times n$ matrix, then $(r+s)A = rA + sA$.
4. If r is a scalar and A and B are $(m \times n)$ matrices, then $r(A+B) = rA + rB$.
5. If A, B, and C are $(m \times n)$, $(n \times p)$, and $(p \times q)$ matrices, respectively, then $(AB)C = A(BC)$.
6. If r and s are scalars, then $r(sA) = (rs)A$.
7. $r(AB) = (rA)B$.
For 8-10, let A and B be $(m \times n)$ matrices and C be an $(n \times p)$ matrix. 8. $(A+B)^T = A^T + B^T$.
9. $(AC)^T = C^T A^T$.
10. $(A^T)^T = A$.

Definition 1.3 *The $(n \times n)$* **identity matrix** *is the matrix*

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} ; \textit{ for example, } I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

I_n is thus the matrix with $1's$ along the main diagonal and $0's$ elsewhere. For any $(n \times n)$ matrix A, $I_n A = A I_n = A$.

Note 1.4 *Given two vectors $\mathbf{v}$ and $\mathbf{w}$ of the same dimension (e.g. both are (1×3) column vectors) thought of as column vectors, note that their dot product can be thought of as the matrix product $\mathbf{v}^T\mathbf{w}$.*

Definition 1.5 *Given a vector $\mathbf{x}$ in $\mathbb{R}^n$ (n-dimensional space), the* **Euclidean length** *(or,* **Euclidean norm***) of $\mathbf{x}$ is $||\mathbf{x}|| = \sqrt{\mathbf{x}^T\mathbf{x}}$.*

2 Linear Independence and Nonsingular Matrices

Definition 2.1 *Let $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_n$ be vectors in $\mathbb{R}^m$. Let $\mathbf{0}$ denote the vector of this dimension whose components are all 0 (note: this is "θ" in the book). A* **linear combination** *of these vectors is an expression of the form $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$ where each a_i is a scalar. If the only such linear combination equal to $\mathbf{0}$ is the one where $a_1 = a_2 = \cdots = a_n = 0$, we say the vectors are* **linearly independent***; otherwise, they are* **linearly dependent***. The set of all linear combinations of these vectors is their* **span***. It's a fact that m linearly independent vectors in $\mathbb{R}^m$ must span all of $\mathbb{R}^m$.*

Theorem 2.2 [Theorem 11, p. 374] *A collection $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ of vectors in $\mathbb{R}^m$ where $p > m$ MUST be linearly dependent.*

Fact 2.3 *Let* $\mathbf{v}_1$*,* $\mathbf{v}_2$*, ..., and* $\mathbf{v}_n$ *be linearly vectors in* $\mathbb{R}^n$ *whose span is all of* $\mathbb{R}^n$*, and let* $\mathbf{b}$ *be another vector in* $\mathbb{R}^n$*. Then, expressing* $\mathbf{b}$ *as a linear combination* $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{b}$ *is equivalent to row reducing to reduced echelon form the matrix*

$$V_b = \left[\begin{array}{ccc|c} | & & | & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n & \mathbf{b} \\ | & & | & | \end{array}\right]$$

to solve for x_1*, ...,* x_n *just as in Section 4.3. Consequently, if* A *is the matrix whose* i*th column is* $\mathbf{v}_i$*, and* $\mathbf{x}$ *is the column vector consisting of the* x_i*, the matrix equation* $A\mathbf{x} = \mathbf{b}$ *has a solution* $\mathbf{x}$ *if and only if* $\mathbf{b}$ *can be expressed as a linear combination of the* $\mathbf{v}_i$*. In particular, if we set* $\mathbf{b} = \mathbf{0}$*, we see that* $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ *is linearly independent if and only if* $x_1 = \cdots = x_n = 0$ *is the only solution to the system of equations represented by* V_b*.*

Procedure 2.4 *To determine whether a collection of vectors* $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\}$ *in* $\mathbb{R}^m$ *is linearly independent, take the matrix* V_b *above with* $\mathbf{b} = 0$ *and row reduce to reduced echelon form. If a matrix* $[B|\mathbf{0}]$ *is obtained, where* $B = I_m$ *or* B *has the form of* I_p *in the top* p *rows and additional rows of* $0'$*s at the bottom, then the vectors are linearly independent. If not, then, you can solve for* x_i *as in Fact 2.3 to show linear dependence.*

Definition 2.5 *An* $(n \times n)$ *matrix* A *is* **nonsingular** *if the only solution to* $A\mathbf{x} = \mathbf{0}$ *(where* $\mathbf{x}$*and*$\mathbf{0}$ *are vectors in* $\mathbb{R}^n$*) is* $\mathbf{x} = 0$*. If* A *is not nonsingular, then we say* A *is* **singular***.*

Theorem 2.6 [Theorem 12, p. 374] *If* A *is an* $(n \times n)$ *matrix that can be expressed as* $[\mathbf{A}_1, ..., \mathbf{A}_n]$*, then* A *is nonsingular if and only if* $\{\mathbf{A}_1, ..., \mathbf{A}_n\}$ *is a linearly independent set.*

Theorem 2.7 [Theorem 13, p. 375] *Let* A *be an* $(n \times n)$ *matrix. The equation* $A\mathbf{x} = \mathbf{b}$ *has a unique solution for every* $(n \times 1)$ *column vector* $\mathbf{b}$ *if and only if* A *is nonsingular.*