1 Consistent systems of linear equations

Recall that our general goal from the previous sections is to find the solution set of a system of equations, and to do this, we perform a sequence of elementary row operations to obtain an **equivalent** system/matrix (meaning one with the same solution set) where the solution set is more obvious. We did this by taking the corresponding augmented matrix and using our row operations to get the matrix to reduced echelon form.

With the same notation as in the previous recitation handout for our $m \times n$ system of equations (m equations in n variables), denote the corresponding $m \times (n+1)$ augmented matrix by $[A \,|\, \mathbf{b}]$ and denote its equivalent reduced echelon form matrix by $[C \,|\, \mathbf{d}]$. Recall that unlike the book, we include a "line of augmentation" to separate the coefficients on the left side from the right side of each equation. **We will abide by this notation for the remainder of this handout.**

Definition 1.1 *Given the matrix $[C \,|\, \mathbf{d}]$ in reduced echelon form, we call an entry in a given column to the left of the line of augmentation a* **leading entry** *if it is the first nonzero entry in its row (note that this is necessarily 1 by the definition of reduced echelon form).*

Note that not every column left of the line of augmentation may have a leading entry (for example, in the augmented matrix on page 327, columns 2,4, and 6 do not have a leading entry; note that the line of augmentation is not drawn on that matrix) and that if there is a leading entry it must be unique.

The use of identifying leading entries is that they allow you to determine independent and dependent variables in the solution set to your system of equations. The **independent variables** (also called **free variables**) are those that can take whatever values one chooses and correspond to the columns without leading entries (for example, in the augmented matrix given on page 327, the free variables would be x_2, x_4, x_6). The **dependent variables** are those whose values depend on the choice of values assigned to the independent variables. Dependent variables correspond to leading terms in a reduced echelon form matrix.

Notation 1.2 *Let r denote the number of nonzero rows in $[C \,|\, \mathbf{d}]$. We call r the* **rank** *of the matrix. Note that necessarily $r \leq n+1$ since there cannot be more leading entries than there are columns and each nonzero row necessarily contains a leading entry.*

Theorem 1.3 [Theorem 3, p.328] *If $[C \,|\, \mathbf{d}]$ represents a consistent system (meaning one with a solution), then $r \leq n$ and the system has $n-r$ free variables. Note that if there are any free variables (i.e. $r < n$), there are necessarily infinitely many solutions. Consequently, the system represented by $[C \,|\, \mathbf{d}]$ is either (a) inconsistent (has no solution), (b) consistent and has infinitely many solutions (since $r < n$), or (c) consistent and has a unique solution (since $r = n$, meaning no free variables).*

Corollary 1.4 [Corollary, p.329] *For a $(m \times n)$ system of linear equations with $m < n$, the system is either inconsistent or has infinitely many solutions (i.e. r=n is impossible).*

Definition 1.5 *Using the same a_{ij} and b_i from the previous handout, an $(m \times n)$ system of linear equations is called* **homogeneous** *if $b_1 = b_2 = \cdots = b_m = 0$, i.e. the column $\mathbf{b}$ in $[A \,|\, \mathbf{b}]$ consists entirely of zeroes.*

Note that a homogeneous system ALWAYS has a solution, the solution $x_1 = \cdots = x_n = 0$, the **trivial solution**. Any other solution is a **nontrivial solution**. Therefore, in light of our corollary, a homogeneous $(m \times n)$ system of linear equations with $m < n$ always has infinitely many solutions.

2 Applications: Networks and Circuits

Procedure 2.1 *To calculate flow in networks, set up an equation for each node in the network, keeping in mind the core principle that* **the flow into a node is equal to the flow out of a node**.

Notation 2.2 *For circuit problems, each gap with a tall line on one side and a shorter line on the other denotes a* **battery**. *This taller line denotes the side of the battery with positive charge, and the shorter line denotes the side with negative charge.* **Current flow always goes from the positive side to the negative side:** *in the event that a problem does not draw arrows to denote current flow for you, use this convention to draw arrows in the proper directions. The letters I_i with arrows denote currents, and* **if they are not provided, you must include one for and immediately adjacent to each resistor in your circuit**. *Resistors (with resistance measured in ohms) are denoted by squiggly lines.*

Our goal is to set up a system of linear equations to solve for the currents, and in order to do that, we need to use the components defined above and a few laws from physics:

Ohm's Law: The voltage drop across a resistor is the product of the current and the resistance.

Kirchhoff's First Law: The sum of the currents flowing into a node is equal to the sum of the currents flowing out.

Kirchhoff's Second Law: The sum of the voltage drops for each resistor around a closed loop (from positive end of a battery to its negative end) is equal to the total voltage drop in the loop.

Generally, Kirchhoff's First Law gets one linear equation for each node (note that these can sometimes be redundant) and Kirchhoff's Second Law used with Ohm's Law gets one linear equation per loop in an electrical network. This resulting system of equations should be sufficient to solve for all the currents in a given electrical network.

3 Matrix Operations

Many of these operations are routinely taught in high school algebra courses, so we just include a couple cautionary statements and facts:

Caution 1: Only matrices of the same size may be added together.

Caution 2: For a product of matrices AB to be defined, you need the number of columns in A to match the number of rows in B. This becomes apparent when you try to multiply because you take dot products of row vectors in A with column vectors in B.

Fact: If A is a $m \times n$ matrix, $\mathbf{x}$ is the column vector consisting of x_1, ..., x_n, and $\mathbf{b}$ is a column vector with m entries, then by matrix multiplication, the vector equation $A\mathbf{x} = \mathbf{b}$ is equivalent to an $m \times n$ system of linear equations in $x_1, ..., x_n$.

Fact: A given $(m \times n)$ matrix A may be written in the form $A = [\mathbf{A_1}, \mathbf{A_2}, ..., \mathbf{A_n}]$, where $\mathbf{A_i}$ denotes the ith column of A considered as a column vector. Given such an A and a $n \times 1$ column vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, the product $A\mathbf{x}$ can be expressed as $A\mathbf{x} = x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_n\mathbf{A}_n$. Moreover, given such an A and a $(l \times m)$ matrix B, we may represent BA as $BA = [B\mathbf{A}_1, B\mathbf{A}_2, ..., B\mathbf{A}_n]$.