1 Potential Functions and Conservative Vector Fields

Definition 1.1 *A* **potential function** *for a vector field* $\mathbf{F} = \langle f, g, h\rangle$ *is a function* φ *such that* $\mathbf{F} = \nabla\varphi$. *A vector field* $\mathbf{F}$ *is* **conservative** *if it has a potential function.*

Procedure 1.2 *Given a vector field* $\mathbf{F} = \langle f, g, h\rangle$, *to find a potential function* φ *for* $\mathbf{F}$:

1. *Take* $\int f dx$. *We get* $\varphi = \left[\int f dx\right] + c(y, z)$, *where* $c(y, z)$ *is a function of* y *and* z, *an "integration constant" for our multivariable function* φ.
2. *Take* φ_y *and compare with* g *(they should be equal) to solve for* $c_y(y, z)$.
3. *Take* $\int c_y(y, z)dy$. *We get that* $c(y, z) = \left[\int c_y(y, z)dy\right] + d(z)$, *where* $d(z)$ *is a function of* z *that we're treating as an "integration constant" for our multivariable function* c.
4. *Take* φ_z *and compare with* h *to solve for* $d'(z)$ *and therefore* $d(z)$ *(up to a constant). Putting all these pieces together completely solves for* φ.

With experience and practice, one often finds that some potential functions are very obvious and easy to find and don't require this full procedure: with an educated guess verified by taking partial derivatives, you can save yourself a fair amount of work in finding potential functions. But what good is knowing a vector field is conservative? What's the point of finding a potential function?

The use of conservative vector fields and their potential functions is that on certain paths you can apply the Fundamental Theorem of Calculus for Line Integrals to them, which saves a *significant* amount of time in computation.

Theorem 1.3 [The Fundamental Theorem of Calculus for Line Integrals] *If* $\mathbf{F}$ *is a conservative vector field on a region* R *in* $\mathbb{R}^3$, *for any piecewise smooth oriented curve* C *in* R *with starting point* A *and ending point* B, *we have*

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \varphi(B) - \varphi(A),$$

where φ *is the potential function for* $\mathbf{F}$.

Some easy consequences of this are the following:

1. Under the hypotheses given in the theorem, any two curves C_1 and C_2 as in the theorem having the same starting and ending points, their integrals are the same. We call this **path independence**.
2. Under the hypotheses given in the theorem, if C is **simple** (meaning it doesn't cross itself) and **closed** (i.e. it starts and ends at the same point, meaning it's a loop), then its integral is 0.

2 Matrices and Systems of Linear Equations

A system of linear equations

$$\begin{cases} a_{11}x_1 + \cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + \cdots a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + \cdots a_{mn}x_n = b_m \end{cases}$$

may be equivalently represented as an augmented matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

and vice-versa. We'll generally be focusing on the augmented matrix since it has an inherently simpler form. Our general goal is to find the solution set of the system of equations, and to do this, we perform a sequence of **elementary row operations** to obtain **equivalent** systems/matrices (meaning one with the same solution set).

Notation 2.1 *The elementary row operations that we may perform to obtain an equivalent system of equations or equivalent matrix are the following. We also include the shorthand we'll use when we perform them.*

1. $R_i \leftrightarrow R_j$ *Switch equations (rows) i and j.*
2. kR_i *Multiply the ith row by $k \neq 0$.*
3. $R_i + kR_j$ *Add k times the jth row to the ith row (get new ith row).*

The equivalent matrix that you generally want to get is either in echelon or reduced echelon form.

Definition 2.2 *An $(m \times n)$ matrix B is in* **echelon form** *if:*

1. *All rows consisting entirely of zeroes are grouped together at the bottom of the matrix.*
2. *In every nonzero row, the first nonzero entry (taken from left to right) is a 1.*
3. *If the $(i+1)$th row has nonzero entries, then the first nonzero entry of that row is in a column to the right of the first nonzero entry in the ith row.*

Heuristically, the nonzero entries should form a staircase-like pattern.

Definition 2.3 *A matrix that is in echelon form is in* **reduced echelon form** *if the first nonzero entry in each row with nonzero entries is the only nonzero entry in its column.*

Here are a few more tips for and observations about what may occur when you row reduce a matrix:

1. A column of zeroes may appear. In that case, the variable corresponding to that column is **free** meaning the other variables may be expressed in term of that variable.
2. A row of zeroes may appear. In that case, the corresponding equation for that row was redundant.
3. A bottom row of $\left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & c \end{array}\right)$ (where c is a constant) means your system of equations is **inconsistent**, meaning it has no solution.
4. In getting to reduced echelon form, try to follow the procedure given on page 318. In short, you should get the desired 1's in your column before getting the desired 0's in the columns. Moreover, you should be treating columns to the left before you treat columns to the right.