

3.2, Problem 15: Evaluate $\int_C (x^2 + y^2) ds$, where C is the circle of radius 4 centered at the origin.

Solution:

Step 1: Parametrize C . We know from Math 1172 how to parametrize a circle of radius 4 centered at the origin: the parametrization is given by $\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle 4 \cos t, 4 \sin t \rangle$, $0 \leq t \leq 2\pi$.

Step 2: Find $|\mathbf{r}'(t)|$. $\mathbf{r}'(t) = \langle -4 \sin t, 4 \cos t \rangle$, so $|\mathbf{r}'(t)| = \sqrt{(-4 \cos t)^2 + (4 \sin t)^2} = 4$, since $\sin^2 t + \cos^2 t = 1$.

Step 3: Evaluate $\int_C (x^2 + y^2) ds$. In general, $\int_C f ds = \int f(x(t), y(t)) \cdot |\mathbf{r}'(t)| dt$ (with numerical integration bounds). Thus, $\int_0^{2\pi} (x^2 + y^2) ds = \int_0^{2\pi} [(4 \cos t)^2 + (4 \sin t)^2] \cdot 4 dt = \int_0^{2\pi} 16 \cdot 4 dt = 128\pi$.

3.2, Problem 37: Given a vector field $\mathbf{F} = \frac{\langle x, y \rangle}{(x^2 + y^2)^{3/2}}$ and a curve C parametrized by $\mathbf{r}(t) = \langle t^2, 3t^2 \rangle$, $1 \leq t \leq 2$, evaluate $\int_C \mathbf{F} \cdot \mathbf{T} ds$.

Solution: Recall in general that given a curve C parametrized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, $a \leq t \leq b$, by definition $\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_a^b \mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t) dt$. **ALSO NOTE** that this same form calculates the work achieved by moving a particle with force \mathbf{F} along the curve C .

So, we compute each individual component of this formula.

$$\mathbf{F}(x(t), y(t)) = \frac{\langle t^2, 3t^2 \rangle}{((t^2)^2 + (3t^2)^2)^{3/2}} = \frac{1}{10^{3/2}} \left\langle \frac{1}{t^4}, \frac{3}{t^4} \right\rangle,$$

and

$$\mathbf{r}'(t) = \langle 2t, 6t \rangle,$$

so

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_1^2 \frac{1}{10^{3/2}} \left\langle \frac{1}{t^4}, \frac{3}{t^4} \right\rangle \cdot \langle 2t, 6t \rangle dt = \frac{1}{10^{3/2}} \int_1^2 \left(\frac{2}{t^3} + \frac{18}{t^3} \right) dt = \frac{3}{40} \sqrt{10}.$$