1 General solutions to homogeneous equations

In this section, we solve

$$y''(t) + p(t)y'(t) + qy(t) = 0 \tag{1}$$

where p and q are scalars.

Consider the **characteristic polynomial** $r^2 + pr + q$. By the quadratic formula, its roots are $r_1 = \frac{-p+\sqrt{p^2-4q}}{2}$ and $r_2 = \frac{-p-\sqrt{p^2-4q}}{2}$. We have three cases:

Case 1: $p^2 - 4q > 0$. In this case, r_1 and r_2 are real numbers and distinct from each other, and the general solution to (1) is $y = c_1e^{r_1t} + c_2e^{r_2t}$, where c_1 and c_2 are arbitrary real numbers.

Case 2: $p^2 - 4q = 0$. In this case, $r_1 = r_2 = -p/2$, a real number, and the general soluiton to (1) is in this case $y = c_1te^{r_1t} + c_2e^{r_1t}$, where c_1 and c_2 are arbitrary real numbers.

Case 3: $p^2 - 4q < 0$. In this case, r_1 and r_2 are complex numbers $a \pm ib$ and the general solution to (1) is in this case $y = c_1e^{at}\cos(bt) + c_2e^{at}\sin(bt)$, where c_1 and c_2 are arbitrary real numbers.

Definition 1.1 *Given a solution $y = c_1\sin(\omega t) + c_2\cos(\omega t)$, its* **amplitude-phase form** *is $y = A\sin(\omega t + \varphi)$, where $A = \sqrt{c_1^2 + c_2^2}$, $\tan(\varphi) = \frac{c_2}{c_2}$ (think of φ as the angle determining the vector $\langle c_1, c_2\rangle$ in the plane, counterclockwise from the positive x-axis).*

Definition 1.2 *For a solution $y(t)$, the* **phase plane** *of $y(t)$ is the parametric curve $r(t) = \langle y(t), y'(t)\rangle$.*

Next we consider another equation, the **Cauchy-Euler Equation**, solved by similar methods:

$$t^2y''(t) + aty'(t) + by(t) = 0 \tag{2}$$

where a and b are real numbers and $t > 0$.

To solve this kind of equation, we consider a *DIFFERENT* **characteristic polynomial** $p(x) = x^2 + (a-1)x + b$. We have 3 cases:

Case 1: If $p(x)$ has two distinct real roots r_1 and r_2, then the general solution is $y = c_1t^{r_1} + c_2t^{r_2}$, where c_1 and c_2 are arbitrary real numbers and $t > 0$.

Case 2: If $p(x)$ has a repeated real root r, then the general solution is $y = c_1t^r + c_2t^r\ln(t)$, where c_1 and c_2 are arbitrary real numbers and $t > 0$.

Case 3: If $p(x)$ has complex roots $\alpha \pm i\beta$, then the general solution is $y = c_1t^\alpha\cos(\beta\ln(t)) + c_2t^\alpha\sin(\beta\ln(t))$, where c_1 and c_2 are arbitrary real numbers and $t > 0$.

2 General solutions to nonhomogeneous equations

In this section, we wish to solve the equation

$$y''(t) + p(t)y'(t) + qy(t) = f(t) \tag{3}$$

where p and q are scalars and $f(t)$ is a function. Per Fact 3 on the handout for Section 5.1, we just need to find one particular solution y_p for (3), and in that case, the general solution to (3) is $y = y_p + y_h$ where y_h is the general solution to the homogeneous equation (directions on how to find this given in the previous section of this handout). Generally, we determine y_p by trial and error: we plug in a trial solution with undetermined coefficients (meaning we don't know what they are; we have to solve for them) and determine the values of the coefficients by plugging in the trial solution and its derivatives into (3). Almost always, the trial solutions are:

f(t)	**trial solution**
degree n polynomial	another degree n polynomial
e^{at}	Ae^{at}
$\sin(bt)$ or $\cos(bt)$	$A\sin(bt) + B\cos(bt)$
$p_n(t)e^{at}$	$q_n(t)e^{at}$
$p_n(t)\sin(bt)$ (or $\cos(bt)$)	$q_n(t)\sin(bt) + r_n(t)\cos(bt)$
$e^{at}\sin(bt)$ (or $\cos(bt)$)	$Ae^{at}\sin(bt) + Be^{at}\cos(bt)$
sum of any of above	sum of corresponding trial solutions

where we adopt the shorthand that $p_n(t)$, $q_n(t)$, and $r_n(t)$ denote degree n polynomials.

CAUTION: Sometimes trial solutions end up being solutions to the homogeneous equation. In that case, multiply the trial solution by t and try again.

Remark 2.1 *Given values for $y(0)$ and $y'(0)$ (we call these* **initial conditions***) and an equation such as (1) and (3) (we call this setup an* **initial value problem** *or IVP), we can find the solution to the IVP by finding the general solution to the equation (as described in this handout) and using the given values of $y(0)$ and $y'(0)$ to solve for c_1 and c_2. There are also corresponding IVP's for the Cauchy-Euler equation (2); in that case, we use $y(1)$ and $y'(1)$.*