1 History and motivation

Theorem 1.1 (Pappus, 300 AD) *Given two lines, each with a trio of points we'll call A, B, C and A', B', C', then the points A'', B'', C'' arising from connecting B with C' and C with B', A with C' and A' with C, and A with B' and B with A', respectively, are colinear.*

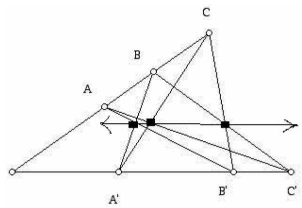

Figure 1: A visualization of Pappus' theorem

Calling this a theorem is a little misleading, because sometimes a pair of the connecting lines considered are parallel, which means the c_i the theorem says exists doesn't exist! It does make sense, though, as we'll see, to say that the c_i is "at infinity." But what does this mean, and why is this a natural thing to consider? To start, let's talk about art in the Renaissance.

The problem posed to artists of the Renaissance was how to represent 3-dimensional objects on a 2-dimensional space. How they accomplished this was with a notion of perspective. Here's how it works: somewhere on the page is a horizontal line we call "the horizon." This is much like how we see things in real life. Also, in real life, bigger things become smaller and wider things become narrower as they get farther away: so, what happens is that parallel lines that aren't parallel to the horizon are forced to meet at the horizon since the space between them must shrink! So, we get this sort of "natural illusion" that parallel lines "meet at infinity," since the horizon is infinitely far out. Looks like that notion of "meeting at infinity" wasn't so unnatural after all!

To illustrate what's going on with this notion of perspective, let's draw a square grid and represent it with this notion of perspective (so, it will look sort of like a road on the page):

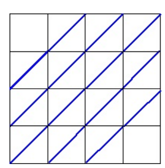

Figure 2: Original square grid with diagonals highlighted

We highlighted the diagonals above since they'll be intersecting at a point on the horizon (we choose it), depicted below, and they'll determine what the rest of the grid looks like from just one square.

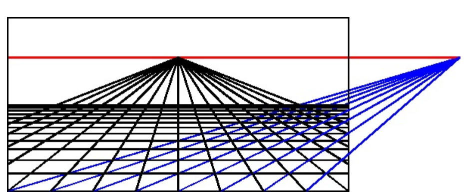

Figure 3: Our square grid drawn in perspective

Now, in order for Pappus' theorem to hold, Gerard Desargues noticed in the 17th century we need every family of parallel lines to meet at a point "at infinity" and for simplicity and consistency, we'll say that these form a "line" that wraps around the plane like a circle: it's what's called a **projective line**: we take an ordinary line and adjoin a "point at infinity" which can be approached from either the positive or negative direction. It's like we're closing up the ordinary line and getting a circle. We want this line at infinity so that 2 things we want hold: any two lines meet somewhere and any 2 points determine a line. If we adjoin the projective line "at infinity" to the ordinary plane, we get the **projective plane**.

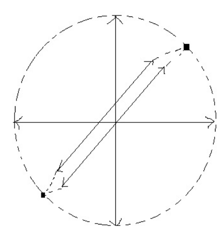

Figure 4: Projective plane drawn as the "completion" of the ordinary plane

Notice in the above drawing that the antipodal points (that is, points on opposite sides) depicted on the circle (actually a projective line) are really the *same* points because 2 lines are only allowed to meet at one point (otherwise, they're the exact same line, which isn't the case). So, what happens is that lines in the projective plane are essentially closed curves (meaning curves that start and end at the same point). For those of you who have taken or are currently enrolled in 145B, what is drawn above should remind you of the following picture.

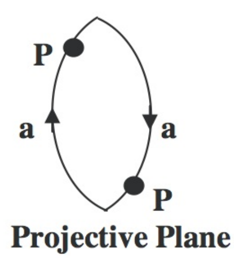

Figure 5: Plane model of $\mathbb{RP}^2$ as seen in beginning algebraic topology (Math 145B at UCR)

For those of you who haven't seen this diagram before, what we do with it is we stretch and twist it (even allowing it to cross over itself, because that's the only way we can make this work) so that the sides labelled "a" are lying on top of each other and have their arrows facing the same way and that the points labelled "P" lie on top of each other. We consider 2 things which are the same up to shrinking, stretching, and twisting **homeomorphic**, and this is the notion of "sameness" that matters to topologists.

Now, with this diagram and the interpretation just described in mind, we can identify the projective plane with a **nonorientable surface** in $\mathbb{R}^3$, meaning there's a path beginning and ending at the same point you can trace along the surface where if we pick an "up direction" normal to the surface, it will change at the end of the path (this is easiest to see with a Möbius strip). This surface we identify with the projective plane in 3-dimensional space is very hard to visualize (a former TA of mine, Edward Burkard, who spoke at the May 20, 2014 UCR Math Club meeting, once described the surface as looking like Pac-Man eating himself).

When Desargues came up with this notion of projective space, though, few mathematicians saw much use for it, so it was largely forgotten about for a couple centuries. It wasn't until the 19th century that projective geometry finally took hold, and this was because a new way to think about it was discovered.

The new view is the following: the projective line is the set of all 1-dimensional subspaces (that is, lines going through the origin) in the plane, and the projective plane is the set of all 1-dimensional subspaces in 3-dimensional space. Similarly, we can define any other n-dimensional projective space, which we denote by $\mathbb{RP}^n$, as the set of all 1-dimensional subspaces of $\mathbb{R}^{n+1}$ (we can generalize this to fields other than the real numbers, but we're not interested in this right now).

Let's see why these 2 different views we discussed actually describe the same thing, in the sense of there being a **homeomorphism** between them: this basically means every point in one corresponds to a point in the other, and neighboring points stay together in this correspondence (though, these "neighborhoods" may stretch or shrink).

First, we'll draw the projective line with this new definition. Then, if we draw in the line $y = 1$, all lines through the origin intersect this at exactly one point (so, we assign each line to that intersection point on the line $y = 1$ and vice-versa), except for x-axis, which is parallel, and we'll say that this is our point at infinity. We see that this correspondence is a homeomorphism (in this imprecise way we defined it) like we wanted, so we're done.

Next, we'll see by very similar reasoning that the two definitions of projective plane given match up in the same way. We'll start by drawing according to the new definition. Then, we add the plane $z = 1$, conduct similar reasoning, and notice that the lines going through the origin lying on the plane $z = 0$ form a projective line "at infinity" for the ordinary plane $z = 1$.

2 Homogeneous Coordinates and Varieties

One helpful way of thinking of this new definition is with **homogeneous coordinates**. The idea is this: our "points" in projective space are 1-dimensional subspaces, so we identify all nonzero vectors in each subspace as the same. What this means is that for any $\lambda \neq 0$, (x_1, x_2, x_3) is the same thing as $(\lambda x_1, \lambda x_2, \lambda x_3)$. We write this equivalence class as the homogeneous coordinates $[x_1 : x_2 : x_3]$. We exclude $\lambda = 0$ since otherwise every point in 3-dimensional space would be considered the same, which we don't want.

What homogeneous coordinates do for us is we can assign "Euclidean coordinates" to "most" points in projective space. For instance, if $z_0 \neq 0$, then we identify $[x_0 : y_0 : z_0]$ with $X = \frac{x_0}{z_0}$ and $Y = \frac{y_0}{z_0}$, which we would treat like ordinary coordinates in Euclidean space. But we can also work backwards from this. For instance, let's consider the parabola $Y = X^2$, or $X^2 - Y = 0$. It's typical to consider a polynomial in some numbers of variables $f(X_1, ..., X_n)$ with coefficients in the real or complex numbers (or any field - if you don't know what a field is, don't worry about it) set equal to 0 and consider all tuples $(a_1, ..., a_n) \in \mathbb{R}^n$ (or in $\mathbb{C}^n$ or whatever the base field is; we'll stick to $\mathbb{R}^n$ for now so we can visualize more easily) : we call the set of solutions $(a_1, ..., a_n)$ to $f(X_1, ..., X_n) = 0$ for any polynomial $f(X_1, ..., X_n) \in k[X_1, ..., X_n]$ (where k is a field, in our case $\mathbb{C}$) a **variety** defined by a single polynomial equation. We can define a variety by multiple polynomial equations as all the tuples satisfying all the given polynomial equations simultaneously, but we aren't going to consider that right now.

In the same way we get Euclidean coordinates from homogeneous coordinates, we can work backwards to get homogeneous coordinates in one dimension higher from Euclidean coordinates.

We can consider what this does to polynomial equations. For instance, let's consider the variety $Y - X^2 = 0$ in the affine plane. If we change to homogeneous coordinates, we notice $Y = \frac{y}{z}$ and $X = \frac{x}{z}$, provided $z \neq 0$, so if we make the substitution, we get a new equation $yz - x^2 = 0$, a variety in 3-dimensional space. This polynomial $f(x, y, z) = yz - x^2$ is special because it is **homogeneous**, meaning $f(\lambda x, \lambda y, \lambda z) = \lambda^i f(x, y, z)$, where i is the degree of the polynomial. What this means is that all terms are the same degree. In a similar way, we can make any polynomial in any number of variables homogeneous by adding in a new variable: we call the resulting variety the **projective closure** of the original variety.

$x^2 = yz$ is actually a surface in 3-dimensional space. Being the "zero locus" of the homogeneous polynomial $x^2 - yz$, it's very easy to graph: given a solution, no matter how we scale it, we still get a solution, i.e. (x_0, y_0, z_0) is a solution if and only if $(\lambda x_0, \lambda y_0, \lambda z_0)$ is a solution for all $\lambda \in \mathbb{R}$ So, if we set $z = 1$, the solutions lying on the plane $z = 1$ define a parabola "$y = x^2$," and scaling these solutions, we get all points on the surface, except the points where $z = 0$. For the solutions where $z = 0$, x must be 0, and y can vary however it pleases; so, the y-axis is included in our solution set. These are all our solutions, so we have all of our graph for $x^2 = yz$. Interestingly, if we intersect this graph with the unit sphere, we get 2 "antipodal" ellipses.

What we've just seen here is that the zero locus of a collection of homogeneous polynomials are **projective varieties** since we can consider the solutions as points with homogeneous coordinates in projective space. The right way to think about this definition is that these are precisely the varieties we can define over projective space, since the only "points" we can plug in (in projective space) have homogeneous coordinates.

Basic algebraic geometry concerns itself with the study of varieties, and one interesting problem considered is the problem of resolutions of singularities. What we mean by this is that some varieties have some points where the "tangent direction" isn't well-defined, like at cusps and nodes. For example, the graph of $y^2 - x^- x^3 = 0$ has this problem. We want to resolve this problem and keep most of the other information we have on these varieties. With curves in 2-dimensional space like the one just given, it's pretty easy to visualize, and we give a visualization of what happens when we "blow up" the curve (remove its singularities) below:

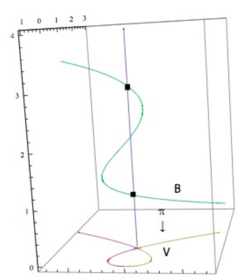

Figure 6: Blow up of $y^2 - x^2 - x^3 = 0$

With projective space introduced in the right way, we can obtain a new variety from the given variety which doesn't have singularities but is "the same" as the original everywhere but where the singularities were, in a sufficient sense. The original UCR Math Club talk finished at this point, but we'll make this more precise in an appendix section. The material from this point on is more advanced and assumes greater knowledge of topology. It's also not as well-prepared or well-written.

3 Appendix on Blow-ups

Let's make all of this more precise. First, we can define the blow-up of $\mathbb{A}_k^n$ (**n-dimensional affine space**, or the space of n-tuples of elements of k, where k is a field) at a point (n-tuple) p, which we can assume to be the origin, O: it's the set of points $(x_1, ..., x_n; [y_1 : ... : y_n]) \in \mathbb{A}^n \times \mathbb{P}^{n-1}$ such that $x_i y_j - y_i x_j = 0$ for all pairs of i and j varying between 1 and n. We'll call this variety B, and it comes packaged with a map $\varphi : B \to \mathbb{A}^n$ defined by $(\{x_i\}, [\{y_i\}]) \mapsto (\{x_i\})$. For all points $p \neq O$, there's a unique point in B that maps to it via φ, for if $p = (a_1, ..., a_n)$, then the "y_i" are given by $y_j = \frac{a_j}{a_i} y_i$. O has a copy of $\mathbb{P}^{n-1}$ associated to it. What this does for us is, since every possible line through the origin in $\mathbb{A}^n$ is associated with a point in $\mathbb{P}^{n-1}$ and all "homogeneous tuples" $[y_1 : ... : y_n]$ satisfy $x_i y_j - y_i x_j = 0$ if $(x_1, ..., x_n)$ is the origin (meaning the origin has a copy of $\mathbb{P}^{n-1}$ attached to it, which we call the **exceptional divisor** of the blowup, and all other points are just associated to themselves), we can separate the lines going through p (the origin) in a very natural way: each line through the origin intersects the exceptional divisor at exactly one point, and hence when we blow up a variety the blown-up variety intersects the exceptional divisor exactly once for every time the original curve passes through the origin. On a given variety, we want to blow-up **singularities**.

Given a variety V in $\mathbb{A}^n$ is defined by polynomials $f_1, ..., f_m$, we say V is **singular** at a point p if the matrix given below has rank less than $\min\{m, n\}$.

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

What this corresponds to is the notion that the "tangent direction" at the point p if you're going along the given curve (we'll call it V) isn't well-defined. We "fix" this by taking the closure of the primage of $V \setminus \{O\}$, $\overline{\varphi^{-1}(V \setminus \{O\})}$, and this resulting object is called the **blow up** of V at O.

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