Procedure 0.1 *Note that to do* **polynomial interpolation** *(that is, determine the polynomial of degree n passing through $n+1$ given points) you should follow the same procedure as for conics in Section 4.3.*

Definition 0.2 *Let $\{t_0, ..., t_n\}$ be a collection of numbers. The $[(n+1)\times(n+1)]$* **Vandermonde matrix** *for $\{t_0, ..., t_n\}$ is the matrix*

$$V_n = \begin{bmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^n \\ 1 & t_1 & t_1^2 & \cdots & t_1^n \\ \vdots & \vdots & & \vdots & \\ 1 & t_n & t_n^2 & \cdots & t_n^n \end{bmatrix}$$

1 Numerical Integration

Let f be a function defined on $[a, b]$, and let $t_1, ..., t_n$ be a collection of numbers in $[a, b]$. Then, there are constants $A_1, ..., A_n$ such that

$$\int_a^b f(t)dt \approx \sum_{i=1}^n A_i f(t_i). \tag{1}$$

In particular, if $p(x)$ is the degree n polynomial obtained by interpolating the points $(t_0, f(t_0)), ..., (t_n, f(t_n))$, then the A_i are chosen so that

$$\int_a^b p(t)dt = \sum_{i=1}^n A_i f(t_i). \tag{2}$$

We find these A_i with the matrix equation

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_0 & t_1 & \cdots & t_n \\ \cdots & \cdots & \cdots & \cdots \\ t_0^n & t_1^n & \cdots & t_n^n \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} \int_a^b t^0 dt \\ \int_a^b t dt \\ \vdots \\ \int_a^b t^n dt \end{bmatrix}$$

Note that $\int_a^b t^i dt = \frac{b^{n+1}-a^{n+1}}{n+1}$. We can therefore equivalently repesent this matrix equation as the augmented matrix

$$\left[V_n^T \;\middle|\; \int_a^b t^i dt \right] = \left[\begin{array}{cccc|c} 1 & 1 & \cdots & 1 & b-a \\ t_0 & t_1 & \cdots & t_n & \frac{b^2-a^2}{2} \\ \cdots & \cdots & \cdots & \cdots & \vdots \\ t_0^n & t_1^n & \cdots & t_n^n & \frac{b^{n+1}-a^{n+1}}{n+1} \end{array}\right]. \tag{3}$$

Row reduce this augmented matrix to reduced echelon form to solve for $A_0, ..., A_n$. Then, substituting these A_i into approximation (1), you obtain your desired approximation of $\int_a^b f(t)dt$.

2 Numerical Differentiation

This is very similar to numerical integration, except that for a differentiable function f on an interval $[a, b]$, some real number c in $[a, b]$, and some collection of real numbers $\{t_0, ..., t_n\}$ in $[a, b]$, you want to find an approximation

$$f'(c) \approx A_0 f(t_0) + \cdots A_n f(t_n). \tag{4}$$

In particular, if $p(x)$ is the degree n polynomial obtained by interpolating the points $(t_0, f(t_0)), ..., (t_n, f(t_n))$, then the A_i are chosen so that

$$p'(c) = A_0 f(t_0) + \cdots A_n f(t_n). \tag{5}$$

We find these A_i by reducing the augmented matrix

$$\left[V_n^T \,\middle|\, \begin{matrix} | \\ \frac{d}{dt}(t^i)|_{t=c} \\ | \end{matrix} \right] = \left[\begin{array}{cccc|c} 1 & 1 & \cdots & 1 & 0 \\ t_0 & t_1 & \cdots & t_n & 1 \\ t_0^2 & t_1^2 & \cdots & t_n^2 & 2c \\ \cdots & \cdots & \cdots & \cdots & \vdots \\ t_0^n & t_1^n & \cdots & t_n^n & nc^{n-1} \end{array} \right], \tag{6}$$

where $0 = \frac{d}{dt}(t^0)|_{t=c}$, $1 = \frac{d}{dt}(t^1)|_{t=c}$, $2c = \frac{d}{dt}(t^2)|_{t=c}$, and so on. Then, substituting the found A_i into approximation (4), you obtain your desired approximation of $f'(c)$.