1 Sets

In math, sets and functions are probably the most important tools available to model real life phenomena and communicate ideas. First, let's give some definitions. Given a set A and an element a, to express "a is an element of A," we write $a \in A$. Similarly, for "a is not an element of A," we write $a \notin A$.

Definition 1.1 *A **set** is a collection of objects, called **elements**.*

Generally, we give sets names (oftentimes, we'll use capital letters), sometimes with special symbols. Some sets which we'll encounter frequently in the course and the associated symbols that denote them are the following:

1. $\mathbb{N}$, the set of natural numbers 1, 2, 3, etc.
2. $\mathbb{Z}$, the set of all integers
3. $\mathbb{Q}$, the set of all rational numbers (fractions of integers)
4. $\mathbb{R}$, the set of all real numbers

Definition 1.2 ***Set builder notation*** *is the convention for building sets by selecting all elements x with a certain property. Generally, to denote the set of all elements x in the set A satisfying a property P, we write*

$$\{x \in A \mid x \text{ has property } P\}.$$

If a set A is not specified, we're considering <u>*all real numbers*</u> *x.*

Example 1.3

$$\{0, 2, 4, 6, 8, 10\} = \{x \mid 0 \leq x \leq 10 \text{ and } x \text{ is even}\}$$

Let $a \in \mathbb{R}$ and $b \in \mathbb{R}$. Take the following as definitions:

1. $[a, b] = \{x \mid a \leq x \leq b\}$
2. $(a, b) = \{x \mid a < x < b\}$
3. $[a, b) = \{x \mid a \leq x < b\}$
4. $(a, b] = \{x \mid a < x \leq b\}$
5. $(a, \infty) = \{x \mid x > a\}$
6. $[a, \infty) = \{x \mid x \geq a\}$
7. $(-\infty, b) = \{x \mid x < b\}$
8. $(-\infty, b] = \{x \mid x \leq b\}$

The moral to this notation is that parentheses mean to exclude the endpoint(s) and square brackets tell you to include the endpoint(s).

Definition 1.4 *An* **ordered pair** *is a pair of numbers (a, b) where if $a \neq b$, then (a, b) and (b, a) are considered distinct. The collection of all ordered pairs of real numbers is denoted by $\mathbb{R}^2$. Given an ordered pair (a, b), its first number a is its* **x-coordinate** *and its second number b is called its* **y-coordinate**. *To every ordered pair of real numbers (a, b) corresponds a point (a, b) in the* **coordinate plane**.

Definition 1.5 *A* **relation** *is a collection of ordered pairs. The* **domain** *of a relation is the collection of x such that (x, y) is in the relation for some y. The* **range** *of a relation is the collection of y such that (x, y) is in the relation for some x.*

Example 1.6 *Consider the relation $R = \{(1, 2), (2, 5), (3, 1)\}$. Then, the domain of R is $\{1, 2, 3\}$ and the range of R is $\{2, 5, 1\}$.*

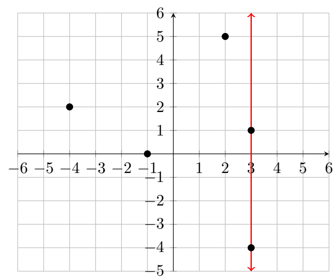
Definition 1.7 *Let R be a relation in x and y. We say that* **y is a function of x** *if for all x in the domain of R there is EXACTLY ONE y such that $(x, y) \in R$. In this case, we say that R* **is the graph of a function** *(oftentimes, the textbook says "R is a function" for short).*

Note 1.8 *It's completely fine to have more than one x correspond to a single y in the range of a relation and still have a function. We still have a function, but the function is not one-to-one and therefore not invertible (we will discuss this later in the course).*

Example 1.9 *Is $R = \{(3, 1), (2, 5), (-4, 2), (-1, 0), (3, -4)\}$ the graph of a function?*

Answer: The number 3 in the domain of R has 2 different numbers (1 and -4) in the range of R corresponding to it, since $(3, 1) \in R$ and $(3, -4) \in R$. So, R CANNOT be the graph of a function, and we say y is NOT a function of x.

Procedure 1.10 [Vertical Line Test] *Given a relation R, plot its points in the coordinate plane. If you can draw a vertical line that passes through 2 points in R, then R is NOT the graph of a function. If you can't draw such a vertical line, then R IS the graph of a function. We call this procedure the* **vertical line test**. *For example, the picture below demonstrates by the vertical line test that the example we just considered is not the graph of a function.*



Fact 1.11 *Relations are often formed as the collection of ordered pairs (x, y) satisfying an equation. If we can get y on its own on one side of the equation defining a relation, then y is a function of x.*

Definition 1.12 *Given two sets A and B, a* ***function*** *$f : A \to B$ is a rule assigning each element x in A to EXACTLY ONE element y in B; in this case, we write $y = f(x)$. Here x is the* **input variable** *from the set A, referred to as the* **domain** *of the function f, and $y = f(x)$ is the* **output** *or* **function value** *in B that x is assigned to, and we call B the* **range** *of the function f. The collection of all ordered pairs $(x, f(x))$ is called the* **graph** *of f.*

Generally, the most useful way to think about functions is through their graphs. In this course, we'll be almost exclusively looking at functions $f : \mathbb{R} \to \mathbb{R}$ with graphs $\{(x, y) \in \mathbb{R}^2 \mid y = f(x)\}$ which we can represent pictorially in the coordinate plane (through extrapolation from a few important points, of course!). It's an important fact that graphs describe functions completely, so in a sense, a function and its graph are the same thing!

Note 1.13 *When a relation R satisfies the condition to be the graph of a function, we may not be given a name for the function that it is the graph of. In that case, we usually just say $y = f(x)$, meaning we just gave our function the name "f." If R is defined by an equation $y =$ _____, then we call the right hand side of the equation "$f(x)$," and f is the name we have given the rule assigning x to $f(x)$.*

Example 1.14 *For the function f (the letter f here is arbitrary) which assigns every real number x to its absolute value $|x|$, we write $f(x) = |x|$ to represent f. This describes the rule completely and concisely! Every real number has an absolute value, so the domain of f is all of $\mathbb{R}$. The absolute value of a number is always greater than or equal to 0, so the range of f is $[0, \infty)$.*

Example 1.15 *By the Fact, the equation $y = 3x^2 + 2x$ defines a function (let's call it f) that assigns each $x \in \mathbb{R}$ to a real number $f(x) = 3x^2 + 2x$. For example, 2 is assigned to $f(2) = 3(2)^2 + 2(2) = 12 + 4 = 16$. As another example,*

$$\begin{aligned}
f(a+h) &= 3(a+h)^2 + 2(a+h) \\
&= 3[(a+h)(a+h)] + 2(a+h) && \text{DON'T FORGET TO FOIL!} \\
&= 3[a^2 + 2ah + h^2] + 2(a+h) \\
&= 3a^2 + 6ah + 3h^2 + 2a + 2h && \text{Distribute}
\end{aligned}$$

Definition 1.16 *Suppose a function is defined by an equation $y = f(x)$. The* **x-intercepts** *are the real number solutions to the equation $f(x) = 0$; on the graph of f, these are the y-coordinates of the points where the graph meets the x-axis (the line $y = 0$). The y-intercept is given by $f(0)$, and it is the y-coordinate of the point where the graph meets the y-axis (the line $x = 0$).*

Theorem 1.17 *The* **Zero Product Property** *states that if a product $ab = 0$, then either $a = 0$ or $b = 0$ (or both!). This is our main tool for determining x-intercepts.*

Example 1.18 *Find the x and y intercepts for $f(x) = x^3 - 2x^2 - x + 2$.*

Solution: The y-intercept is $f(0) = 0^3 - 2 \cdot (0)^2 - (0) + 2 = 2$ The x-intercepts are the solutions to the equation $x^3 - 2x^2 - x + 2 = 0$.

$$\begin{aligned}
x^3 - 2x^2 - x + 2 &= 0 \\
x^2(x-2) - 1(x-2) &= 0 && \textit{Factor by grouping} \\
(x^2 - 1)(x-2) &= 0 \\
(x-1)(x+1)(x-2) &= 0 && \textit{Difference of squares rule: } a^2 - b^2 = (a-b)(a+b) \\
x - 1 = 0, \quad \text{or} \quad x + 1 = 0, \quad \text{or} \quad x - 2 &= 0 && \textit{Zero Product Property}
\end{aligned}$$

$$x = 1 \quad \text{or} \quad x = -1 \quad \text{or} \quad x = 2.$$

2 Concluding Remark/Reminder

Please supplement this handout by reading all of Sections 2.1 and 2.2 and your notes taken in lecture. This is a supplement to the textbook's coverage, not a substitute. I've put some content from Chapter 1 in this handout as well, and you should read the chapter if you're baffled by any of what I've discussed in the first few sections of this handout. I've written this to ensure all of us are on the same page as far as notation goes - if we don't use the same notation and terminology, communicating with each other can be very confusing!