1 Introduction to Circulation and Flux

We'll begin by giving intuitive definitions and then formal definitions (i.e. how to compute them) for circulation and flux for a vector field $\mathbf{F}$ on/across a curve C in the xy-plane.

Intuitively, **circulation** of a vector field $\mathbf{F}$ on a curve C is how much $\mathbf{F}$ points in the same direction of C's orientation: agreement counts as positive circulation and disagreement counts as negative circulation.

Intuitively, **flux** of a vector field $\mathbf{F}$ across a curve C is how much $\mathbf{F}$ points in the same direction as the (outward facing) normal vector to C. So, everywhere $\mathbf{F}$ points outward is a positive contribution to flux; everywhere it points inward counts as a negative contribution to flux.

Now, we give the formal definitions:

Definition 1.1 *The* **circulation** *of a vector field* $\mathbf{F}$ *on a piecewise smooth oriented curve* C *parametrized by* $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$, *is given by* $\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_a^b \mathbf{F}(x(t), y(t))\cdot\mathbf{r}'(t)dt$.

Definition 1.2 *The* **flux** *(also called* **outward flux***) of a continuous vector field* $\mathbf{F} = \langle f(x,y), g(x,y)\rangle$ *across a piecewise smooth oriented curve* C *parametrized by* $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$, *is given by* $\int_C \mathbf{F}\cdot\mathbf{n}ds = \int_a^b [f(x(t), y(t))y'(t) - g(x(t), y(t))x'(t)]dt$.

For both of these definitions, it suffices to compute just the right hand side of the equations given in the definitions (the integrals with respect to t). With these points in mind, let's do a couple circulation and flux problems.

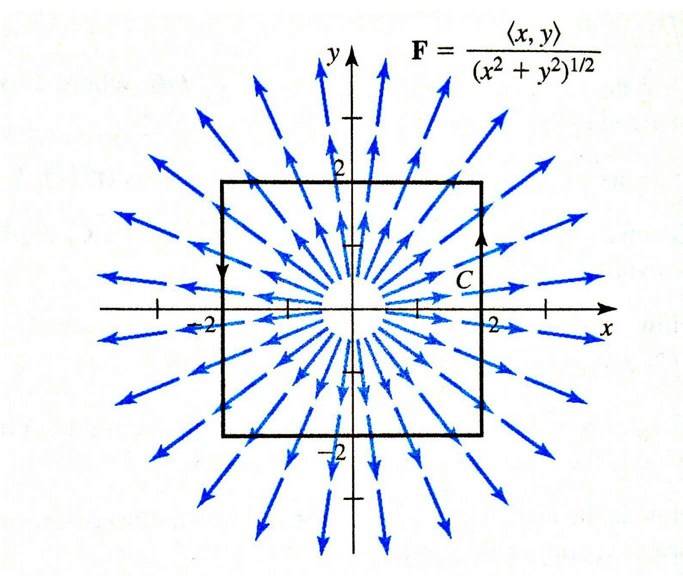

Figure 1: Vector field for 3.2 problems 48 and 50

3.2, Problem 48: Let C be the boundary of the square with vertices $(\pm2, \pm2)$, traversed counterclockwise, as pictured above. For the above vector field $\mathbf{F}$, based on the picture above, make a conjecture about whether the circulation of $\mathbf{F}$ on C is positive, negative, or zero. Then, compute the circulation and interpret the result.

Solution:
Notice that the vector field (being radial) is completely symmetric: it points clockwise just as much as it points counterclockwise. As our intuitive definiton would dictate, then, the circulation of $\mathbf{F}$ on C should be 0.

Now, we back this conjecture by computation. Since we're doing line integrals, we start by parametrizing the boundary by four paths: $\mathbf{r_1}(t) = \langle 2, -2+4t\rangle$ (top edge C_1), $\mathbf{r_2}(t) = \langle 2-4t, 2\rangle$ (right edge C_2), $\mathbf{r_3}(t) = \langle -2, 2-4t\rangle$ (left edge C_3), and $\mathbf{r_4}(t) = \langle -2+4t, -2\rangle$ (bottom edge C_4), all with $0 \le t \le 1$. Then, $\mathbf{r_1'}(t) = \langle 0, 4\rangle$,$\mathbf{r_2'}(t) = \langle -4, 0\rangle$, $\mathbf{r_3'}(t) = \langle 0, -4\rangle$, and $\mathbf{r_4'}(t) = \langle 4, 0\rangle$. Then, $\int_C \mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t)dt = \sum_{i=1}^4 \int_{C_i} \mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t)dt = \int_0^1 \frac{1}{\sqrt{8-16t+16t^2}} \cdot \langle 2, -2+4t\rangle \cdot \langle 0, 4\rangle dt + \int_0^1 \frac{1}{\sqrt{8-16t+16t^2}} \cdot \langle 2-4t, 2\rangle \cdot \langle -4, 0\rangle dt + \int_0^1 \frac{1}{\sqrt{8-16t+16t^2}} \cdot \langle -2, 2-4t\rangle \cdot \langle 0, -4\rangle dt + \int_0^1 \frac{1}{\sqrt{8-16t+16t^2}} \langle -2+4t, -2\rangle \cdot \langle 4, 0\rangle dt = \int_0^1 \frac{64t-32}{\sqrt{8-16t+16t^2}} dt = 0$, as desired.

3.2, Problem 50: Let C be the boundary of the square with vertices $(\pm 2, \pm 2)$, traversed counterclockwise, as pictured on the first page. For the above vector field $\mathbf{F}$, based on the picture above, make a conjecture about whether the outward flux of $\mathbf{F}$ across C is positive, negative, or zero. Then, compute the flux and interpret the result.

Solution: Notice that the vector field always points outward, so our intuitive definition would dictate that our outward flux should be positive.

Now, we back this conjecture by computation. We parametrize the boundary just as in Problem 48, using the same $\mathbf{r_i}(t) = \langle x(t), y(t)\rangle$ there, and so our $\mathbf{r_i'}(t)$ are therefore the same. Then, appealing to our formal definition $\int_C \mathbf{F} \cdot \mathbf{n} ds = \sum_{i=1}^4 \int_{C_i} \mathbf{F} \cdot \mathbf{n} ds = \sum_{i=1}^4 \int_0^1 [\frac{x(t)}{\sqrt{x(t)^2+y(t)^2}} \cdot y'(t) - \frac{y(t)}{\sqrt{x(t)^2+y(t)^2}} \cdot x'(t)]dt = \sum_{i=1}^4 \int_0^1 \frac{1}{\sqrt{x(t)^2+y(t)^2}}[x(t) \cdot y'(t) - y(t) \cdot x'(t)]dt = \int_0^1 \frac{1}{\sqrt{8-16t+16t^2}}(2 \cdot 4 - 0)dt + \int_0^1 \frac{1}{\sqrt{8-16t+16t^2}}(0 + 2 \cdot 4)dt + \int_0^1 \frac{1}{\sqrt{8-16t+16t^2}}((-2)(-4) - 0)dt + \int_0^1 \frac{1}{\sqrt{8-16t+16t^2}}(0 + 2 \cdot 4)dt = \int_0^1 \frac{32}{\sqrt{8-16t+16t^2}} dt = 8\ln(2\sqrt{2}+3) \approx 14.1$, as desired.