# 1 Fourier series

**Definition 1.1** *Let $f$ be a piecewise continuous function on the interval $[-L, L]$. The* **Fourier series** *of $f$ is given by*

$$S_\infty f(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\},$$

*where $a_n$ and $b_n$ are given by the* **Euler formulas**

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \ \ \text{for } n = 0, 1, 2, ...$$

*and*

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \ \ \text{for } n = 1, 2, 3, ...$$

**Definition 1.2** *Given a piecewise continuous function $f$ on the interval $[0, L]$, the* **Fourier cosine series** *of $f$ is*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

*where for each $n \geq 0$*

$$a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx,$$

*and the* **Fourier sine series** *is*

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

*where for each $n \geq 1$*

$$b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

# 2 The Heat Equation

## 2.1 Insulated ends at $0°$C, no external heat

For the heat-flow problem

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2},\ 0 < x < L,\ t > 0$$
$$u(0, t) = u(L, t) = 0,\ t > 0$$
$$u(x, 0) = f(x),\ 0 < x < L$$

the solution is

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\beta(n\pi/L)^2 t}, \tag{1}$$

where

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \tag{2}$$

is the Fourier sine series for $u(x, 0)$

### 2.2 Zero net heat flow out the ends, no external heat

For the heat-flow problem

$$\begin{gathered}\frac{\partial u}{\partial t} = \beta\frac{\partial^2 u}{\partial x^2},\ 0 < x < L,\ t > 0\\ \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0,\ t > 0\\ u(x,0) = f(x),\ 0 < x < L\end{gathered}$$

the solution is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\beta(n\pi/L)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

to the original I-BVP, where $a_n = \dfrac{2}{L}\displaystyle\int_0^L u(x,0)\cos\left(\frac{n\pi x}{L}\right)dx$, i.e. we compute the Fourier cosine series for $u(x,0)$.

### 2.3 Wire ends at different temperatures, no external heat

For the heat-flow problem

$$\begin{gathered}\frac{\partial u}{\partial t} = \beta\frac{\partial^2 u}{\partial x^2},\ 0 < x < L,\ t > 0\\ u(0,t) = U_1,\ u(L,t) = U_2,\ t > 0\\ u(x,0) = f(x),\ 0 < x < L\end{gathered}$$

, where $U_1$ and $U_2$ are both constants (not both 0), the solution is

$$u(x,t) = v(x) + \sum_{n=1}^{\infty} c_n e^{-\beta(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right),$$

where $v(x) = U_1 + \frac{U_2 - U_1}{L}x$ and the $c_n$ are the Fourier sine coefficients for $w(x,0) := u(x,0) - v(x)$.

### 2.4 Wire ends at different temperatures, with external heat

For the heat-flow problem

$$\begin{gathered}\frac{\partial u}{\partial t} = \beta\frac{\partial^2 u}{\partial x^2} + P(x),\ 0 < x < L,\ t > 0\\ u(0,t) = U_1,\ u(L,t) = U_2,\ t > 0\\ u(x,0) = f(x),\ 0 < x < L\end{gathered}$$

, where $U_1$ and $U_2$ are both constants (not both 0) and $P(x)$ is the external heat source, the solution is

$$u(x,t) = v(x) + \sum_{n=1}^{\infty} c_n e^{-\beta(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right),$$

where

$$v(x) = \left[U_2 - U_1 + \int_0^L \left(\int_0^z \frac{1}{\beta}P(s)ds\right)dz\right]\frac{x}{L} + U_1 - \int_0^x \left(\int_0^z \frac{1}{\beta}P(s)ds\right)dz$$

and the $c_n$ are the Fourier sine coefficients for $w(x,0) := u(x,0) - v(x)$.

# 3 The wave equation

## 3.1 Basic scenario

For the vibrating string problem

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < L,\ t > 0 \\ u(0,t) = u(L,t) = 0,\ t > 0 \\ u(x,0) = f(x),\ 0 < x < L \\ \frac{\partial u}{\partial t}(x,0) = g(x),\ 0 < x < L\end{gathered}$$

the solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi\alpha}{L}t\right)\sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi\alpha}{L}t\right)\sin\left(\frac{n\pi x}{L}\right),$$

where

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \tag{3}$$

is the Fourier sine series for $f(x) = u(x,0)$ and

$$\sum_{n=1}^{\infty} b_n \left(\frac{n\pi\alpha}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \tag{4}$$

is the Fourier sine series for $\frac{\partial u}{\partial t}(x,0) = g(x)$.

## 3.2 String of infinite length

For the vibrating string problem

$$\begin{gathered}\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2},\ -\infty < x < -\infty,\ t > 0 \\ u(x,0) = f(x),\ -\infty < x < -\infty \\ \frac{\partial u}{\partial t}(x,0) = g(x),\ -\infty < x < -\infty\end{gathered}$$

the solution is

$$u(x,t) = \frac{1}{2}[f(x+\alpha t) + f(x-\alpha t)] + \frac{1}{2\alpha}\int_{x-\alpha t}^{x+\alpha t} g(s)ds \tag{5}$$