Strategies for Choosing New Variables

Sometimes a change of variables simplifies the integrand but leads to an awkward region of integration. Conversely, the new region of integration may be simplified at the expense of additional complications in the integrand. Here are a few suggestions for finding new variables of integration. The observations are made with respect to double integrals, but they also apply to triple integrals. As before, R is the original region of integration in the xy-plane and S is the new region in the uv-plane.

1. **Aim for simple regions of integration in the uv-plane** The new region of integration in the uv-plane should be as simple as possible. Double integrals are easiest to evaluate over rectangular regions with sides parallel to the coordinate axes.

➤ Inverting the transformation means solving for x and y in terms of u and v, or vice versa.

2. **Is $(x, y) \to (u, v)$ or $(u, v) \to (x, y)$ better?** For some problems it is easiest to write (x, y) as functions of (u, v); in other cases, the opposite is true. Depending on the problem, inverting the transformation (finding relations that go in the opposite direction) may be easy, difficult, or impossible.
   - If you know (x, y) in terms of (u, v) (that is, $x = g(u, v)$ and $y = h(u, v)$), then computing the Jacobian is straightforward, as is sketching the region R given the region S. However, the transformation must be inverted to determine the shape of S.
   - If you know (u, v) in terms of (x, y) (that is, $u = G(x, y)$ and $v = H(x, y)$), then sketching the region S is straightforward. However, the transformation must be inverted to compute the Jacobian.
3. **Let the integrand suggest new variables** New variables are often chosen to simplify the integrand. For example, the integrand $\sqrt{\dfrac{x - y}{x + y}}$ calls for new variables $u = x - y$ and $v = x + y$ (or $u = x + y, v = x - y$). There is, however, no guarantee that this change of variables will simplify the region of integration. In cases in which only one combination of variables appears, let one new variable be that combination and let the other new variable be unchanged. For example, if the integrand is $(x + 4y)^{3/2}$, try letting $u = x + 4y$ and $v = y$.
4. **Let the region suggest new variables** Example 5 illustrates an ideal situation. It occurs when the region R is bounded by two pairs of "parallel" curves in the families $g(x, y) = C_1$ and $h(x, y) = C_2$ (Figure 14.84). In this case, the new region of integration is a rectangle $S = \{(u, v): a_1 \le u \le a_2, b_1 \le v \le b_2\}$, where $u = g(x, y)$ and $v = h(x, y)$.

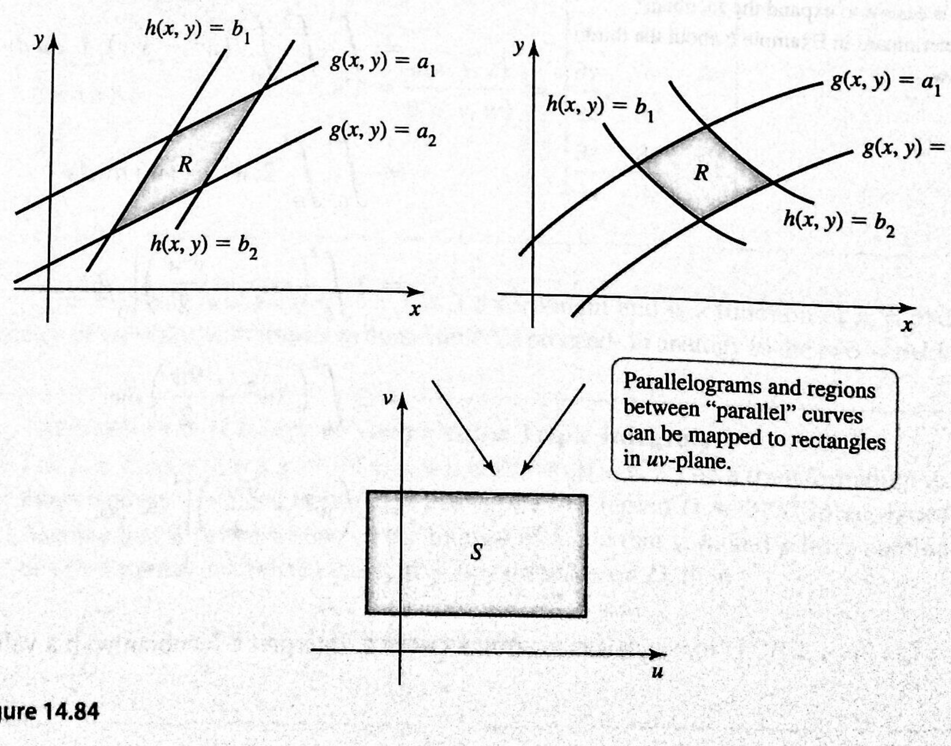

Figure 14.84