# 1 More general heat equation scenarios

We're still modelling heat flow on a wire, but we change the hypotheses on the ends of the wire; they will not necessarily be at $0°C$ at all times.

**Scenario 1:** Zero net heat flow in and out the ends of the wire, i.e. $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$ for all $t > 0$. Consequently, using separation of variables with $u(x,t) = X(x)T(t)$, we get $X'(0) = X'(L) = 0$, getting us solutions $X_n(x) = a_n \cos\left(\frac{n\pi x}{L}\right)$ to the ODE with $X(x)$ and thus a formal series solution

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\beta(n\pi/L)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

to the original I-BVP, where $a_n = \frac{2}{L}\int_0^L u(x,0)\cos\left(\frac{n\pi x}{L}\right) dx$.

**Thus, the $a_n$ above are derived from the Fourier cosine series for $u(x,0)$.**

**Scenario 2:** $u(0,t) = U_1$ and $u(L,t) = U_2$, where $U_1$ and $U_2$ are both constants, not both 0; in this case, we say the heat flow problem is **nonhomogeneous**. In this case, we have an allowed assumption:

**Allowed assumption:** When given nonhomogeneous boundary conditions, we may decompose $u(x,t)$ as $u(x,t) = v(x) + w(x,t)$ where $v(0) = U_1$, $v(L) = U_2$, and $w$ and all its partial derivatives tend to 0 as $t \to \infty$. We call $v$ the **steady-state solution** and $w$ the **transient solution**.

**The significance of this assumption** is that $w(x,t)$ then satisfies homogeneous boundary conditions, allowing us to reduce to the homogeneous case with a little bit of work. When $\frac{\partial u}{\partial t} = \beta\frac{\partial^2 u}{\partial x^2}$, we have $\frac{\partial w}{\partial t} = \beta v''(x) + \beta\frac{\partial^2 w}{\partial x^2}$. Since the partial derivatives of $w$ go to 0 as $t \to \infty$, we have $v''(x) = 0$, meaning by the boundary conditions on $v$ that $v(x) = U_1 + \frac{U_2 - U_1}{L}x$. Thus, $w(x,0) = u(x,0) - v(x)$. This means we just have to find the Fourier sine series of $w(x,0)$ (with coefficients $c_n$) to get the full formal solution

$$u(x,t) = v(x) + \sum_{n=1}^{\infty} c_n e^{-\beta(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right).$$

Again, **the $c_n$ are the coefficients from the Fourier sine series for $w(x,0)$.**

**Scenario 3:** There's also an external heat source present that's independent of time, given by $P(x)$. In this case, the heat equation becomes $\frac{\partial u}{\partial t} = \beta\frac{\partial^2 u}{\partial x^2} + P(x)$, and instead of $v''(x) = 0$ as in the immediately preceding paragraph, we have $v''(x) = -\frac{1}{\beta}P(x)$ (note that we still have $v(0) = U_1$ and $v(L) = U_2$; recall that we had by definition $u(0,t) = U_1$ and $u(L,t) = U_2$), so then

$$v(x) = \left[U_2 - U_1 + \int_0^L \left(\int_0^z \frac{1}{\beta}P(s)ds\right)dz\right]\frac{x}{L} + U_1 - \int_0^x \left(\int_0^z \frac{1}{\beta}P(s)ds\right)dz.$$

This weird formula is derived on pages 520 and 521 of your textbook. From there, one proceeds just as in Scenario 2 (just with this different $v(x)$).

# 2 The wave equation

For the vibrating string problem

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < L,\ t > 0$$
$$u(0,t) = u(L,t) = 0,\ t > 0$$
$$u(x,0) = f(x),\ 0 < x < L$$
$$\frac{\partial u}{\partial t}(x,0) = g(x),\ 0 < x < L$$

the solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi\alpha}{L}t\right) \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi\alpha}{L}t\right) \sin\left(\frac{n\pi x}{L}\right),$$

where

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \tag{1}$$

is the Fourier sine series for $f(x) = u(x,0)$ and

$$\sum_{n=1}^{\infty} b_n \left(\frac{n\pi\alpha}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \tag{2}$$

is the Fourier sine series for $\frac{\partial u}{\partial t}(x,0) = g(x)$.

**Remark 2.1** *Notice that each mode* $\left[a_n \cos\left(\frac{n\pi\alpha}{L}t\right) + b_n \sin\left(\frac{n\pi\alpha}{L}t\right)\right] \sin\left(\frac{n\pi x}{L}\right)$ *represents a family of waves on* $[0, L]$ *with amplitudes varying with choice of* $t$*. However, with each choice of* $t$*, at the point* $x = L/n$ *(and some other points), since* $\sin(n\pi x/L) = 0$*, the point stays fixed for all* $t$*; we call these points* **nodes***.*

## 2.1 New scenario: with time dependent forcing

We consider the wave equation with a time-dependent forcing term $h(x,t)$ and obtain a new equation

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + h(x,t). \tag{3}$$

We suppose that we can find decompositions $u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{L}\right)$ and $h(x,t) = \sum_{n=1}^{\infty} h_n(t) \sin\left(\frac{n\pi x}{L}\right)$, where the $u_n(t)$ are to be found by solving the ODE's below and the $h_n(t)$ are given by $h_n(t) = \frac{2}{L}\int_0^L h(x,t) \sin\left(\frac{n\pi x}{L}\right) dx$. Then, (1) becomes

$$\sum_{n=1}^{\infty} \left[u_n''(t) + \left(\frac{n\pi\alpha}{L}\right)^2 u_n(t)\right] \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} h_n(t) \sin\left(\frac{n\pi x}{L}\right),$$

meaning by comparing terms we have for each $n$

$$u_n''(t) + \left(\frac{n\pi\alpha}{L}\right)^2 u_n(t) = h_n(t). \tag{4}$$

By using variation of parameters on (2) (don't worry about what that means since you didn't learn it), we get

$$u_n(t) = a_n \cos\left(\frac{n\pi\alpha}{L}t\right) + b_n \sin\left(\frac{n\pi\alpha}{L}t\right) + \frac{L}{n\pi\alpha}\int_0^t h_n(s) \sin\left(\frac{n\pi\alpha}{L}(t-s)\right) ds,$$

where $a_n$ and $b_n$ are chosen such that $u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$ and $\frac{\partial u}{\partial t}(x,0) = g(x) = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi\alpha}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$.

## 2.2 d'Alembert's solution to the wave equation

We now deal with the scenario that **our string has infinite length**. Using the change of variables $\psi = x + \alpha t$ and $\eta = x - \alpha t$, since then $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \psi^2} + 2\frac{\partial^2 u}{\partial \psi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$ and $\frac{\partial^2 u}{\partial t^2} = \alpha^2 \left( \frac{\partial^2 u}{\partial \psi^2} - 2\frac{\partial^2 u}{\partial \psi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \right)$, we see the wave equation becomes $\frac{\partial^2 u}{\partial \psi \partial \eta} = 0$, so then $u(\psi, \eta) = A(\psi) + B(\eta)$ for some functions $A$ and $B$ that are twice differentiable and to be determined.

**Finding A and B such that** $u(x,t) = A(x+\alpha t) + B(x - \alpha t)$:

We first notice that $u(x,0) = f(x)$ becomes $(*)\quad A(x) + B(x) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$ becomes $\alpha A'(x) - \alpha B'(x) = g(x)$, so then by integrating $(**)\quad A(x) - B(x) = \frac{1}{\alpha}\int_{x_0}^{x} g(s)ds + C$, where $x_0$ and $C$ are arbitrary constants. By taking linear combinations, we have $A(x) = (1/2)f(x) + \frac{1}{2\alpha}\int_{x_0}^{x} g(s)ds + C/2$ and $B(x) = 1/2 f(x) - \frac{1}{2\alpha}\int_{x_0}^{x} g(s)ds - C/2$. Thus, we have

$$u(x,t) = \frac{1}{2}[f(x+\alpha t) + f(x-\alpha t)] + \frac{1}{2\alpha}\int_{x-\alpha t}^{x+\alpha t} g(s)ds \tag{5}$$

The above centered equation is d'Alembert's solution. **KNOW THIS.**

## 2.3 Traveling waves

Let $h$ be a function on the real numbers, represented by a wave graphically (trig function or trigonometric series). For each $t$, $h(x+\alpha t)$ represents a function in $x$, and its graph is $h(x)$ shifted left by $\alpha t$. Letting $t \to \infty$, the wave goes further and further left, and we say $h(x+\alpha t)$ is a **traveling wave** moving to the left with speed $\alpha$. Similarly, $h(x - \alpha t)$ is a traveling wave moving right with speed $\alpha$. Thus, in our solution (3) above, $u(x,t)$ is the sum of the waves $(1/2)f(x+\alpha t)$ and $(1/2)f(x-\alpha t)$; also, notice, that the waves are superimposed at time $t = 0$ and move away from each other with speed $2\alpha$ as $t$ increases.