1 Fourier sine and cosine series

Definition 1.1 *Given a function f defined on $(0, L)$, the* **odd 2L-periodic extension** *is*

$$f_o(x) = \begin{cases} f(x) & \text{if } 0 < x < L \\ -f(-x) & \text{if } -L < x < 0, \end{cases}$$

and the **even 2L-periodic extension** *is*

$$f_e(x) = \begin{cases} f(x) & \text{if } 0 < x < L \\ f(-x) & \text{if } -L < x < 0. \end{cases}$$

Definition 1.2 *Given a piecewise continuous function f on the interval $[0, L]$, the* **Fourier cosine series** *of f is*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where for each $n \geq 0$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx,$$

and the **Fourier sine series** *is*

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where for each $n \geq 1$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

Note that these are, respectively, the Fourier series expansions for f_e and f_o.

The relevance of this to the heat-flow problem on a wire with $0°C$ ends is that you just need to find the Fourier sine series for $u(x, 0) = f(x)$, and its coefficients c_n will be the coefficients in the series solution for $u(x, t)$. Therefore, for the heat-flow problem

$$\begin{gathered} \frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2},\ 0 < x < L,\ t > 0 \\ u(0, t) = u(L, t) = 0,\ t > 0 \\ u(x, 0) = f(x),\ 0 < x < L \end{gathered}$$

the solution is

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\beta(n\pi/L)^2 t}, \tag{1}$$

where

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \tag{2}$$

is the Fourier sine series for $u(x, 0)$