1 The one-dimensional heat equation

We consider a thin wire along a line segment beginning at $x = 0$ and ending at $x = L$, where each x has a circular cross section of area a (same area for all x). Let ρ be the density of the wire at each point x (meaning the wire has constant density), c be the specific heat capacity of the wire, and k be the thermal conductivity of the wire. Let $Q(x, t)$ be the **energy rate density** at the point x at time t, which is defined to be the amount of external heat being introduced to the wire at the point x at time t. At a fixed point x and time t, let $u(x, t)$ denote the temperature at the point x and time t. Moreover, assume that $u(0, t) = u(L, t) = 0^\circ C$ for all t and that no heat enters or leaves through the sides (where $x = 0$ and $x = L$).

Then, $u(x, t)$ is the solution to the following partial differential equation, called the **one-dimensional heat equation**:

$$\frac{\partial u}{\partial t}(x,t) = \beta \cdot \frac{\partial^2 u}{\partial x^2}(x,t) + P(x,t) \tag{1}$$

where $\beta = \frac{k}{c\rho}$, $P(x,t) = \frac{Q(x,t)}{c\rho}$, $0 < x < L$, and $t > 0$. The conditions $u(0, t) = u(L, t) = 0^\circ C$ for all t are called the **boundary conditions**, and if we additionally impose on u the **initial condition**

$$u(x, 0) = f(x) \quad \text{for all x } 0 < x < L,$$

where $f(x)$ is just some given function of x (the initial temperature distribution), then we have an **initial-boundary value problem** (abbreviated I-BVP) with a unique solution $u(x, t)$.

This generalizes to higher dimensions by replacing $\frac{\partial^2 u}{\partial x^2}$ with $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ or $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$, where in both cases Δu is called the **Laplacian** of u. A related equation that is also common, called **Laplace's equation**, is $\Delta u = 0$.

In Section 6.2, we solve the one-dimensional heat equation with $P(x, t) = 0$ (i.e. no external heat source) by using a method called **separation of variables**, meaning we assume that a solution $u(x, t)$ is a product of a function $X(x)$ of x alone and a function $T(t)$ of t alone, i.e. $u(x, t) = X(x)T(t)$, and then we figure out what $X(x)$ and $T(t)$ can be. The beauty of this method is that it reduces solving a partial differential equation (PDE) to solving two ordinary differential equations (ODE's). Plugging this $u(x, t) = X(x)T(t)$ into equation (1) with the additional assumption $P(x, t)$ is identically 0, we get

$$X(x)T'(t) = \beta X''(x)T(t)$$

and consequently two related ODE's

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(L) = 0, \quad \text{and}$$

$$T'(t) + \lambda\beta T(t) = 0,$$

where λ (called an **eigenvalue**), by calculations done in the texbook, must be of the form $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ (where any positive integer n works). Therefore, a solution to the I-BVP must be of the form

$$u_n(x,t) = c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\beta(n\pi/L)^2 t}$$

(called an **eigenfunction** of the problem) or a potentially infinite linear combination of these, namely

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\beta(n\pi/L)^2 t}, \tag{2}$$

called a **Fourier sine series**, which only works provided that the series and its first 2 derivatives converge.

Procedure: In this case, we have $u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$, so if $f(x) = u(x,0)$ is already given to you in this form (a sum of sine terms), all you have to do is plug the c_n in that given (potentially finite) series for $f(x)$ into the general form given in (2).

2 The vibrating string problem

Consider a string going from $x = 0$ to $x = L$ that's perfectly flexible, has constant linear density, and has constant tension. Suppose no external forces act on the string and gravity is negligible. If $u(x,t)$ is the displacement of the string (where > 0 is up and < 0 is down) at any point x ($0 \le x \le L$) and time $t > 0$, we wish to solve the I-BVP given by

1. $\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$, $0 \le x \le L$, $t > 0$,
2. $u(0,t) = u(L,t) = 0$, for all $t \ge 0$,
3. $u(x,0) = f(x)$, $0 \le x \le L$,
4. $\frac{\partial u}{\partial t}(x,0) = g(x)$, $0 \le x \le L$,
5. $f(0) = f(L) = 0$, and $g(0) = g(L) = 0$,

where α^2 is the ratio of the tension to the linear density of the string.

Applying separation of variables to this I-BVP like with the heat equation, finding appropriate λ, $T(t)$, and $X(t)$ (this is all done in detail in the textbook), we get the **formal general solution** (meaning, this solution may or may not converge, and we're disregarding this question of convergence for the moment in the treatment of this infinite sum)

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi\alpha}{L}t\right) + b_n \sin\left(\frac{n\pi\alpha}{L}t\right)\right] \sin\left(\frac{n\pi x}{L}\right).$$

In the language of this general solution, bullet 3 above becomes

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = f(x),\ 0 \le x \le L$$

and 4 becomes

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \frac{n\pi\alpha}{L} \sin\left(\frac{n\pi x}{L}\right) = g(x),\ 0 \le x \le L.$$

Thus, to find the series for u, we have to find the Fourier sine series expansions for f and g as above.

WARNING: Just because we find formulas for a_n and b_n doesn't mean the series above that they're involved in converge (unless the relevant series only involve finitely many terms)! Hence, this is why we say our solutions are **formal**.