1 Oscillator equations

We consider a block with mass m suspended from a spring and its motion deviating from **equilibrium position**, which is the position the spring is stretched to by the object and gravity. The (vertical) position is given by $y(t)$ where $y = 0$ at equilibrium, $y < 0$ when the spring is compressed, and $y > 0$ when the spring is stretched.

Newton's law: $my'' = F_s + F_d + F_{ext}$, where

(a) $F_s = -ky$ is the **restoring force** of the spring and k is solved for by **Hooke's Law**:

$$mg = k \cdot [\text{distance spring is stretched by gravity and the object}],$$

where $g = 9.8\,\frac{m}{s}$

(b) $F_d = -cy'$ is the **damping force** and c is a given **damping constant**

(c) F_{ext} is the sum of the **external forces**; they depend only on t.

Taking the resulting equation and dividing by m, we get our **main equation**

$$y'' + by' + \omega_0^2 y = f(t), \tag{1}$$

where $b = \frac{c}{m}$, $\omega_0^2 = \frac{k}{m}$, and $f(t) = \frac{F_{ext}}{m}$.

If $c = b = 0$, then the motion is **undamped**; otherwise, the motion is **damped**. If $F_{ext} = f = 0$, then we have **free oscillations**; otherwise, we have **forced oscillations**.

1.1 Notes on forced undamped oscillations

We generally assume $F_{ext} = F_0 \cos(\omega t)$ or $F_{ext} = F_0 \sin(\omega t)$, where F_0 and ω are constants.

1. If $\omega \neq \omega_0$, $y(0) = y'(0) = 0$, and $F_{ext} = F_0 \cos(\omega t)$, then the solution to (1) is

$$y = \left[\frac{2F_0}{m\omega_0^2 - m\omega^2} \sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)\right] \left\{\sin\left(\frac{\omega_0 + \omega}{2} t\right)\right\}.$$

   If $\omega \approx \omega_0$, **beats** occur, meaning the larger wave in [ ] encloses the smaller wave in { }. This is also called **amplitude modulation** in circuits.

2. If $\omega = \omega_0$ and $F_{ext} = F_0 \cos(\omega t)$, we're in the "trial particular solution fails" case, so the general solution is

$$y = c_1 \sin(\omega_0 t) + c_2 \cos(\omega_0 t) + \left[\frac{F_0}{2m\omega_0} t \sin(\omega_0 t)\right],$$

   where the portion in [ ] contributes **resonance**, meaning the amplitude of that portion increases with t.

1.2 Notes on damped oscillations

First, note that the characteristic polynomial has roots $r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4\omega_0^2}}{2}$.

1. If $b^2 > 4\omega_0^2$, then $y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, where $r_1 < 0$ and $r_2 < 0$, so we have **overdamping**, meaning that the damping suppresses ALL oscillations.

2. If $b^2 = 4\omega_0^2$, we have a double root $r_1 = -b/2$, so $y_h = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$, where $r_1 < 0$, meaning we have **critical damping** (the damping suppresses MOST oscillations).

3. If $b^2 < 4\omega_0^2$, we have complex roots $r_1, r_2 = -\frac{b}{2} \pm ai$, and $y_h = e^{-bt/2}(c_1 \sin(at) + c_2 \cos(at))$, so **underdamping** occurs, meaning the solutions are oscillatory with decaying amplitudes.

4. In the forced (meaning nonhomogeneous) case when $\omega \neq \omega_0$, the solution is $y = y_h + y_p$ where y_h is as in 1.-3. immediately above and $y_p = A\sin(\omega t) + B\cos(\omega t)$. Since $y_h \to 0$ as $t \to \infty$, y_h is called the **transient solution**, and since consequently $y \to y_p$ as $t \to \infty$, y_p is called the **steady state solution**.

2 Electrical circuits

Given a circuit with

1. resistor with resistance of R ohms,

2. capacitor with capacitance of C farads,

3. inductor coil with inductance of L henries, and

4. voltage source (i.e. a battery; also called **applied voltage**) of $E(t)$ volts,

we generally are tasked to find the **current** of $I(t)$ amperes or the charge of $Q(t)$ coulombs; these are related by $I(t) = \frac{dQ}{dt}$. Typically, this takes the form of the initial value problem

$$LI'' + RI' + \frac{1}{C}I = E'(t) \tag{2}$$

with initial conditions $Q(0) = 0$ and $I(0) = 0$, and **we must deduce** $I'(0) = \frac{E(0)}{L}$. This is derived from the corresponding initial value problem (with same initial conditions) for the charge, given by **Kirchhoff's voltage law**:

$$LQ'' + RQ' + \frac{1}{C}Q = E(t). \tag{3}$$

These problems are solved just like oscillator problems otherwise.