1 Curl and Divergence

Definition 1.1 *Let* $\mathbf{F} = \langle f, g, h\rangle$ *be a differentiable vector field defined on a region* D *of* $\mathbb{R}^3$*. Then, the* **divergence** *of* $\mathbf{F}$ *on* D *is*

$$\text{div } \mathbf{F} := \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h\rangle = f_x + g_y + h_z,$$

where $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle$ *is the* **del operator***. If* $\text{div } \mathbf{F} = 0$*, we say that* $\mathbf{F}$ *is* **source free.**

Note that these definitions (divergence and source free) completely agrees with their 2D analogues in 15.4.

Theorem 1.2 *Suppose that* $\mathbf{F}$ *is a radial vector field, i.e. if* $\mathbf{r} = \langle x, y, z\rangle$*, then for some real number* p*,* $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^p} = \frac{\langle x,y,z\rangle}{(x^2+y^2+z^2)^{p/2}}$*, then* $\text{div } \mathbf{F} = \frac{3-p}{|\mathbf{r}|^p}$.

Theorem 1.3 *Let* $\mathbf{F} = \langle f, g, h\rangle$ *be a differentiable vector field defined on a region* D *of* $\mathbb{R}^3$*. Then, the* **curl** *of* $\mathbf{F}$ *on* D *is*

$$\text{curl } \mathbf{F} := \nabla \times \mathbf{F} = \langle h_y - g_z, f_z - h_x, g_x - f_y\rangle.$$

If $\text{curl } \mathbf{F} = 0$*, then we say* $\mathbf{F}$ *is* **irrotational**.

Note that $g_x - f_y$ is the 2D curl as defined in section 15.4. Therefore, if we fix a point (a, b, c), $[g_x - f_y](a, b, c)$, measures the rotation of $\mathbf{F}$ at the point (a, b, c) in the plane $z = c$. Similarly, $[h_y - g_z](a, b, c)$ measures the rotation of $\mathbf{F}$ in the plane $x = a$ at (a, b, c), and $[f_z - h_x](a, b, c)$ measures the rotation of $\mathbf{F}$ in the plane $y = b$ at the point (a, b, c). In other words, evaluated at the point $P(a, b, c)$, $(h_y - g_z)$ gives the rotation at P of $\mathbf{F}$ about the line $\mathbf{r}(t) = \langle t, b, c\rangle$ (parallel to $\mathbf{i} = \langle 1, 0, 0\rangle$), $(f_z - h_x)$ gives the rotation at P of $\mathbf{F}$ about the line $\mathbf{r}(t) = \langle a, t, c\rangle$ (parallel to $\mathbf{j} = \langle 0, 1, 0\rangle$), and $(g_x - f_y)$ gives the rotation at P of $\mathbf{F}$ about the line $\mathbf{r}(t) = \langle a, b, t\rangle$ (parallel to $\mathbf{k} = \langle 0, 0, 1\rangle$).

Definition 1.4 *Let* $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ *be a nonzero constant vector, and let* $\mathbf{r} = \langle x, y, z\rangle$*, per usual. If* $\mathbf{a}$ *is considered a position vector, it defines a line through the origin, the* **axis of rotation** *for the vector field* $\mathbf{F}$ *defined by* $\mathbf{F} = \mathbf{a} \times \mathbf{r} = \langle a_2z - a_3y, a_3x - a_1z, a_1y - a_2x\rangle$*. We call* $\mathbf{F}$ *the* **general rotation field** *about* $\mathbf{a}$*.*

Notice that if $a_1 = a_2 = 0$ and $a_3 = 1$, then we obtain the familiar 2D rotation field $\langle -y, x\rangle$ in the xy-plane, and at each plane $z = c$, the exact same 2D rotation field appears. In the same way, if $\mathbf{a}$ is a general position vector, $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ will represent a similar 2D rotation field in each plane perpendicular to $\mathbf{a}$.

Theorem 1.5 *Let* $\mathbf{a}$ *be a nonzero position vector and* $\mathbf{F}$ *be its rotation field. Then,*
(1) $\nabla \cdot \mathbf{F} = 0$.
(2) $\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = 2\mathbf{a}$.
(3) *If* $\mathbf{F}$ *represents velocity (i.e.* $\mathbf{F}$ *is a* **velocity field***), then the* **constant angular speed** *of* $\mathbf{F}$ *is*

$$\omega := |\mathbf{a}| = \frac{1}{2}|\nabla \times \mathbf{F}|.$$

Theorem 1.6 *Let* $\mathbf{F}$ *and* $\mathbf{G}$ *be differentiable vector fields in* $\mathbb{R}^3$*, let* c *be a real number, and let* u *be a differentiable real-valued function. Then,*
(1) $\text{curl } (\mathbf{F} + \mathbf{G}) = \text{curl } \mathbf{F} + \text{curl } \mathbf{G}$*,*
(2) $\text{div } (\mathbf{F} + \mathbf{G}) = \text{div } \mathbf{F} + \text{div } \mathbf{G}$*,*
(3) $\text{curl } (c\mathbf{F}) = c \cdot \text{curl } \mathbf{F}$*,*
(4) $\text{div } (c\mathbf{F}) = c \cdot \text{div } \mathbf{F}$*, and*
(5) $\text{curl } (u\mathbf{F}) = (\nabla u) \cdot \mathbf{F} + u(\text{curl } \mathbf{F})$*, where* $\cdot$ *represents the dot product and the latter multiplication is scalar.*

Theorem 1.7 *Let* $\mathbf{F}$ *be a conservative vector field on an open region* D *of* $\mathbb{R}^3$*. Let* φ *be the potential function for* $\mathbf{F}$ *and suppose that* φ *has continuous second partial derivatives. Then,* $\nabla \times \mathbf{F} = 0$*, i.e.* $\text{curl}(\mathbf{F}) = \mathbf{0}$*, and* $\mathbf{F}$ *is therefore irrotational.*

Theorem 1.8 *Let* $\mathbf{F} = \langle f, g, h\rangle$ *be a vector field on* $\mathbb{R}^3$ *such that* f*,* g*, and* h *each have continuous second partial derivatives. Then,* $\nabla \cdot (\nabla \times \mathbf{F}) = 0$*, i.e.* $\text{div}(\text{curl } \mathbf{F}) = 0$.

2 Parametrizing surfaces

Definition 2.1 *A* **closed and bounded surface** *is a collection of points S in $\mathbb{R}^3$ which may be described as the image of rectangle $R = \{(u,v) : a \le u \le b, c \le v \le d\}$ under a mapping of the form $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$. Thus, S is the collection of all points $\{(x,y,z) : (x,y,z) = \mathbf{r}(u,v) \text{ for some } (u,v) \text{ in } R\}$, and the collection of vectors $\{\mathbf{r}(u,v)\}$ traces out all points on S. A surface may be unbounded as well by considering either $-\infty < u < \infty$ and/or $-\infty < v < \infty$.*

Some parametric descriptions of surfaces you should remember are the following:

1. The **cylinder** $x^2+y^2 = a^2$, $0 \le z \le h$. This is described parametrically as $\mathbf{r}(u,v) = \langle a\cos u, a\sin u, v\rangle$, where $0 \le u \le 2\pi$ and $0 \le v \le h$.

2. The **cone** of maximal radius a at $z = h$ and vertex at the origin. This is described parametrically as $\mathbf{r}(u,v) = \langle \frac{av}{h}\cos u, \frac{av}{h}\sin u, v\rangle$, where $0 \le u \le 2\pi$ and $0 \le v \le h$.

3. The **sphere** $x^2+y^2+z^2 = a^2$.This is described parametrically as $\mathbf{r}(u,v) = \langle a\sin u\cos v, a\sin u\sin v, a\cos u\rangle$, where $0 \le u \le \pi$ and $0 \le v \le 2\pi$.

4. The **plane** $ax + by + cz = d$. Since $z = (1/c)(d - ax - by)$, this is described parametrically as $\mathbf{r}(u,v) = \langle u, v, (1/c)(d - au - bv)\rangle$, where $-\infty < u < \infty$ and $-\infty < v < \infty$.

Tip 2.2 *For other surfaces, try to think in terms of polar and spherical coordinates and consider the range of r, θ and z values or ρ, φ, and θ values taken. You should also get a relationship between these that allows you to eliminate one of those variables (setting it equal to an expression of the others). These should help you get a description of your surface in 2 variables, the desired parametric description.*

3 Surface Integrals

On a surface S parametrized by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, $a \le u \le b$ and $c \le v \le d$ (call this rectangle in the uv-plane R), the tangent vector to S corresponding to a positive change in u with v fixed is $\mathbf{t}_u = \langle x_u, y_u, z_u\rangle$, and the tangent vector to S corresponding to a positive change in v with u fixed is $\mathbf{t}_v = \langle x_v, y_v, z_v\rangle$. Their cross product $\mathbf{t}_u \times \mathbf{t}_v$ is a vector normal to S. Moreover, since in small neighborhoods surfaces are nearly planes (tangent plane approximation) if the change in u is Δu and the change in v is Δv, since the area of the parallelogram spanned by $\Delta u \mathbf{t}_u$ and $\Delta v \mathbf{t}_v$ is $|(\Delta u \mathbf{t}_u) \times (\Delta v \mathbf{t}_v)| = |\mathbf{t}_u \times \mathbf{t}_\mathbf{v}|\Delta u \Delta v$. Therefore, "adding up these tiny rectangles" approximating S (using limits $\Delta u \to 0$ and $\Delta v \to 0$), have the following definition.

Definition 3.1 *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous scalar value function, and let S be as above. Suppose S is* **smooth** *(meaning x, y, and z are smooth functions of u and v), and suppose $\mathbf{t}_u$ and $\mathbf{t}_v$ are continuous on $R = [a,b] \times [c,d]$ in the uv-plane and $\mathbf{t}_u \times \mathbf{t}_v$ is nonzero on R. Then, the* **surface integral** *of f over S is*

$$\iint_S f(x,y,z)dS := \iint_R f(x(u,v), y(u,v), z(u,v))|\mathbf{t}_u \times \mathbf{t}_v|\, dA,$$

and if $f \equiv 1$, then the above integral is the surface area of S. Moreover, if S is given explicitly by $z = g(x,y)$ where the pairs (x,y) vary over a region Rin the xy-plane, then the surface integral of f over S is

$$\iint_R f(x,y,g(x,y))\sqrt{g_x^2 + g_y^2 + 1}\, dA.$$

We can find the average value of a function f on S just like in previous sections, and we can integrate a density function over a surface to obtain the mass of the surface.

Definition 3.2 *A surface is* **orientable** *if the normal vectors $\mathbf{t}_u \times \mathbf{t}_v$ vary continuously over the surface, which in particular means the direction the normal vectors point ("up") is well-defined everywhere (an example where this doesn't occur is the Mobius strip). If the direction of the normal vector is determined everywhere on S, we say S is* **oriented**. *It's typical convention for us to have surfaces oriented so that normal vectors point in the* <u>*outward*</u> *direction.*

Since the unit normal vector to S as above is $\mathbf{n} = \frac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|}$, if $\mathbf{F} = \langle f, g, h \rangle$ is a continuous vector field defined on a region containing S,

$$\mathbf{F} \cdot \mathbf{n} \, dS = \mathbf{F} \cdot \frac{\mathbf{t}_u \times \mathbf{t}_v}{|\mathbf{t}_u \times \mathbf{t}_v|} |\mathbf{t}_u \times \mathbf{t}_v| \, dA = \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) \, dA,$$

so we have the following definition.

Definition 3.3 *Let $\mathbf{F}$ be a continuous vector field in a region in $\mathbb{R}^3$ containing a smooth oriented surface S parametrized by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ ((u,v) in a region R), then*

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS := \iint_R \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) \, dA,$$

where $\mathbf{t}_u$ and $\mathbf{t}_v$ are as above and continuous on R and $\mathbf{n}$ is nonzero on R and consistent with the orientation of S. Moreover, if S is defined explicitly by $z = s(x,y)$ for (x,y) in a region R in the xy-plane, then

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_R (-f s_x - g s_y + h) \, dA.$$

Note that the integrals in the above definition measure the **flux** of $\mathbf{F}$ across the surface S. This typically applies when $\mathbf{F}$ has a physical interpretation, i.e. the flow of a fluid or transport of a substance across as S.

4 Stokes' Theorem

Theorem 4.1 [Stokes' Theorem] *Let S be a smooth oriented surface in $\mathbb{R}^3$ with a smooth closed boundary C whose orientation is consistent with that of S. Assume that $\mathbf{F} = \langle f, g, h \rangle$ is a vector field whose components have continuous first partial derivatives on S. Then,*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS,$$

where $\mathbf{n}$ is the unit vector normal to S determined by the orientation of S.

Note 4.2 []

1. *C is oriented counterclockwise if and only if $\mathbf{n}$ is always pointing upward on S.*
2. *You can think of Green's Theorem (circulation form) as a special version of Stokes' Theorem where S lies in the xy plane, so the integral over S is just the usual double integral (" $(u,v) = (x,y)$ and $R = S$"; i.e. S doesn't need to be parametrized) and $\mathbf{n} = \langle 0, 0, 1 \rangle$.*
3. *If $\mathbf{F}$ is conservative, all integrals above are zero because* curl $\mathbf{F} = \mathbf{0}$.
4. *Stokes' Theorem can either be used as a shortcut to evaluate a line integral as a surface integral instead (avoiding parametrizing different parts separately) OR as a shortcut to evaluate a surface integral as a line integral instead (avoiding parametrizing difficult surfaces).*
5. *If S_1 and S_2 are two different oriented surfaces with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively, and common boundary curve C oriented counterclockwise, then Stokes' Theorem says*

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1 \, dS = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, dS$$

   for any vector field $\mathbf{F}$.
6. *If S is a* **closed** *oriented surface (there is no "boundary curve"), then $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = 0$.*
7. *As a consequence of Stokes' Theorem, if* curl $\mathbf{F} = \mathbf{0}$ *everywhere on an open simply connected region D in $\mathbb{R}^3$, then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all simple closed smooth curves C in D and $\mathbf{F}$ is conservative on D.*

5 Divergence Theorem

Naturally extending the flux form of Green's Theorem to surfaces instead of curves, we obtain the following.

Theorem 5.1 [Divergence Theorem] *Let* $\mathbf{F}$ *be a vector field whose components have continuous first partial derivatives on a connected and simply connected region* D*, and let* S *be the closed oriented surface enclosing* D *with outward unit normal vector* $\mathbf{n}$*. Then,*

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV.$$

Thus, the net divergence in a solid is the same as the flux across the surface which bounds it.

Note 5.2 []
1. *The pure rotational vector fields* $\mathbf{F} = \mathbf{a} \times \langle x, y, z \rangle$*, where* $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ *is a fixed vector field, ALWAYS have zero divergence, meaning their flux across a closed oriented surface* S *will always be zero.*
2. *If computing the flux across a surface with surface integrals is arduous (e.g. must parametrize six faces to a cube), the Divergence Theorem can make that ugly sum of surface integrals an easier triple integral.*
3. *If* $\mathbf{v}$ *is the velocity field of a material and* ρ *is its density, then vector field* $\mathbf{F} = \rho\mathbf{v}$ *describes the* **mass transport** *of the material* (*units: mass*/(area · time))*, meaning* $\mathbf{F}$ *gives the mass of the material flowing past a point per unit of surface area per unit of time. Multiplying this by surface area, we obtain flux. The book uses this to motivate the derivation of the Divergence Theorem.*

Theorem 5.3 *Suppose a vector field* $\mathbf{F}$ *satisfies all conditions for the Divergence Theorem on a region* D *enclosed between two oriented closed surfaces* S_1 *and* S_2 *with outward normal vectors* $\mathbf{n}_1$ *and* $\mathbf{n}_2$*, respectively and where*S_1 *lies within* S_2*. Then,*

$$\iiint_D \nabla \cdot \mathbf{F} dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS \; - \; \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS,$$

where $S = S_1 \cup S_2$ *is the entire boundary of* D*.*

Note 5.4 *This version of the Divergence Theorem is applicable to vector fields not differentiable at the origin (e.g. radial vector fields).*

Theorem 5.5 [Gauss' Law]
1. *By the inverse square law, a charge* Q *generated at a point (we'll let it be the origin) creates an electrical field* $\mathbf{E}$ *given by* $\mathbf{E}(x, y, z) = \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{r}}{|\mathbf{r}|^3}$*, where* $\mathbf{r} = \langle x, y, z \rangle$ *and* ϵ_0 *is a physical constant* (*the* **permittivity of free space**)*. Since the flux of* $\frac{\mathbf{r}}{|\mathbf{r}|^3}$ *across any surface that encloses the origin is* 4π (**this is a VERY important property of this vector field**)*, we have*

$$\iint_S \mathbf{E} \cdot \mathbf{n}\, dS = \frac{Q}{\epsilon_0},$$

for any surface S *enclosing a charge* Q *at the origin. Shifting to any point, the computations stay the same, so we may assume* Q *is at any point.*

2. *If instead of a point charge* Q *we have a variable charge density* $q(x, y, z)$ *defined on a region* D *enclosed by a surface* S*, we have* $Q = \iiint_D q(x, y, z) dV$*, so*

$$\iint_S \mathbf{E} \cdot \mathbf{n}\, dS = \frac{1}{\epsilon_0} \iiint_D q(x, y, z)\, dV.$$

3. *If* T *is a temperature distribution in a solid region* D *enclosed by a surface* S*, the vector field* $\mathbf{F} = -k\nabla T$ *is the heat flow vector field of* T*. If* $q(x, y, z)$ *represents the sources of heat within* D*, by Gauss' Law*

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -k \iint_S \nabla T \cdot \mathbf{n}\, dS = \iiint_D q(x, y, z)\, dV.$$